Math 2001. Lecture 32.

04. 13. 2021.

Last time:

more induction: $\left\{ \begin{array}{l} \text{the fundamental thm of arithmetic} \\ \\ \text{Fibonacci numbers} \quad F_1 = 1,\ F_2 = 1,\ F_n = F_{n-1} + F_{n-2} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall n > 2. \end{array} \right.$

Today:

- More on the Fibonacci sequence

- Worksheet on induction / strong induction.

1. More on the Fibonacci sequence

Recall from last time: $\forall\, n \geq 1$, $\quad F_{n+1}^2 - F_{n+1}F_n - F_n^2 \overset{(*)}{=} (-1)^n$.

We derive a consequence:

divide by F_n^2

Facts: (a) $F_n > 0 \;\forall n \geq 1$, $\; F_{n+1} > F_n \;\forall n \geq 2$.

$$\frac{F_{n+1}^2}{F_n^2} - \frac{F_{n+1}F_n}{F_n^2} - \frac{F_n^2}{F_n^2} = \frac{(-1)^n}{F_n^2}$$

i.e.,
$$\left(\frac{F_{n+1}}{F_n}\right)^2 - \frac{F_{n+1}}{F_n} - 1 = \frac{(-1)^n}{F_n^2}$$

(b) Let $r_n = \dfrac{F_{n+1}}{F_n}$. Then $\lim\limits_{n\to\infty} r_n$ exists.

Q: What's $x := \lim r_n$?

A: Since $r_n^2 - r_n - 1 = \frac{(-1)}{F_n^2} \to 0$

We have $\lim_{n\to\infty} (r_n^2 - r_n - 1) = 0$

so $\lim_{n\to\infty} (r_n^2) - \lim_{n\to\infty} r_n - 1 = 0$

$$\left(\lim_{n\to\infty} r_n\right)^2 - \lim_{n\to\infty} r_n - 1 = 0$$

$$x^2 - x - 1 = 0$$

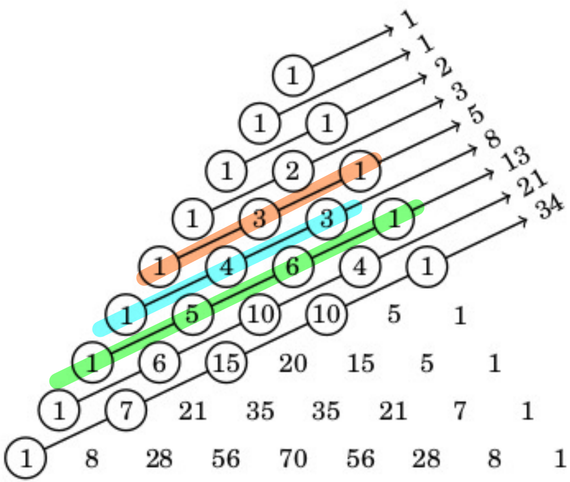
$$x = \frac{+1 \pm \sqrt{5}}{2}.$$

Since $r_n > 0 \ \forall n$, we must have $x > 0$, so $x = \frac{1+\sqrt{5}}{2}$ = "the golden ratio Φ".

We have proven that $\boxed{\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi}$.

One more inductive proof:

10.29. "Entries along the diagonals of Pascal's triangle sum to Fibonacci numbers."



Q1: How do we make the statement precise

: $\binom{4}{0}+\binom{3}{1}+\binom{2}{2}=F_5$

: $\binom{5}{0}+\binom{4}{1}+\binom{3}{2}=F_6$

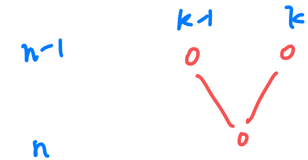
: $\binom{6}{0}+\binom{5}{1}+\binom{4}{2}+\binom{3}{3}=F_7$

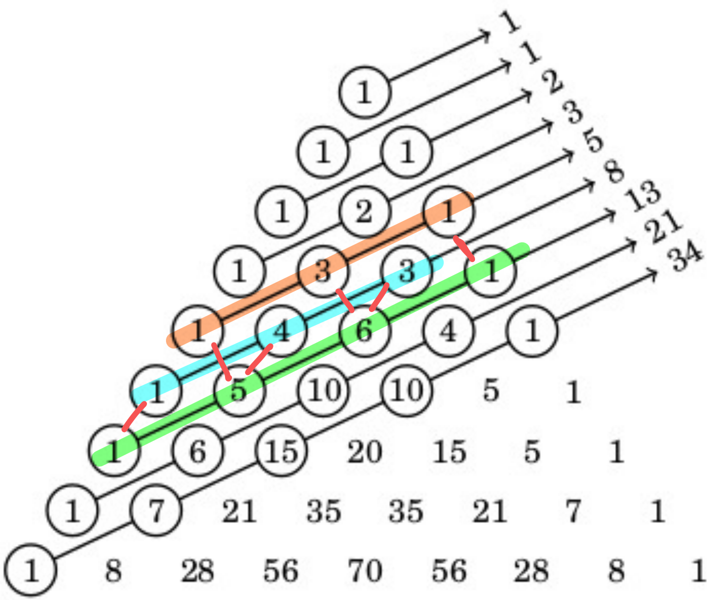
A1: The precise statement is:

(*) if n is
- even, w/ $n=2k$, then $\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\dots+\binom{k}{k}=F_{n+1}$
- odd, w/ $n=2k+1$, then $\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\dots+\binom{k+1}{k}=F_{n+1}$.

Q2: How do we prove (✱) ?

A2: Use strong induction and the (recursive) fact that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$





← representative example:

Sum of + sum of

Sum of

*Ex: Can you turn these ideas into a proof ?

2. Worksheet