Math 2001. Lecture 30.

04. 08. 2022.

recursion is key!

Last time:

- more induction proofs
- strong (mathematical) induction: to prove S_n for all $n \geq 1$, it suffices to (1) prove the base case/cases (2) prove that $\forall k$, $S_1 \wedge S_2 \wedge \cdots \wedge S_k \Rightarrow S_{k+1}$.

Today:

- finishing the stamp example, more strong induction proofs
- A problem on Fibonnaci numbers.

1. Strong induction examples

(a) Show that any postage of 8 cents or more can be formed by combining 3-cent and 5-cent stamps.

Pf: It suffices to show that for any integer $n \geq 8$, $\underbrace{\text{we can write } n \text{ as a sum of 3s and 5s}}_{S_n}$. We do so by induction on n.

Base cases: if $n = 8$, then we can write n as $n = 8 = 3+5$.

" $n = 9$, " " " " " $n = 9 = 3+3+3$.

" $n = 10$, " " " " " $n = 10 = 5+5$.

So S_n holds for all $8 \leq n \leq 10$.

Inductive step: Suppose S_n holds for all n in $8, 9, 10, \ldots, k$ for some $k \geq 10$. We want to show that S_{k+1} must then also hold. (Note that $k+1 \geq 11$.)

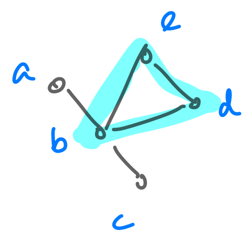
To prove S_{k+1}, we want to show that $(k+1)$ can be written as a sum of 3s and 5s. Since $k+1 \geq 11$, we have $(k+1)-3 \geq 8$, so by the inductive hypothesis, we can write $(k+1)-3$ as a sum of 3s and 5s, say it's a sum of x 3s and y 5s. It follows that $(k+1) = \left[(k+1)-3\right] + 3 = (x+1)\cdot 3 + y \cdot 5$.

So S_{k+1} holds.

It follows that S_n holds for all $n \geq 8$, as desired.

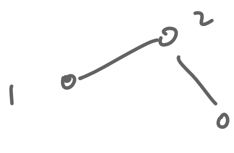
(b) (Trees) Background: An (undirected) graph, abstractly, is the data of a set V of "vertices" and a set E of "edges" where each edge contains two vertices.

e.g.



$\rightarrow \quad V = \{a, b, c, d, e\}$

$E = \{\{a, b\}, \{b, c\}, \{b, d\}, \{b, e\}, \{d, e\}\}$

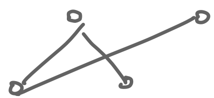


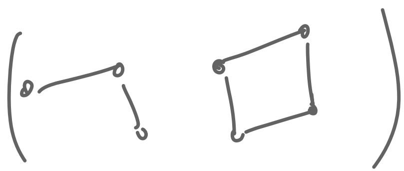
$\leftarrow \quad V = \{1, 2, 3\}, \quad E = \{\{1, 2\}, \{2, 3\}\}$

A cycle in a graph $G = (V, E)$ is a sequence $v_1, v_2, \cdots, v_k$ s.t. $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_1\}$ are all edges. (E.g. $b - d - e - b$ in the first example.)

A graph $G=(V,E)$ is connected if there is a path from x to y $\forall x,y \in V$.

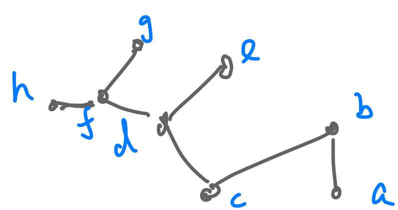
$\exists$ a seq. $x=v_1, v_2, \dots, v_n=y$
s.t. $\{v_i, v_{i+1}\} \in E$ $\forall 1 \le i \le n-1$.

(1)  is connected.

(2)  is not connected.

A tree is a connected acyclic graph.
no cycle

e.g.



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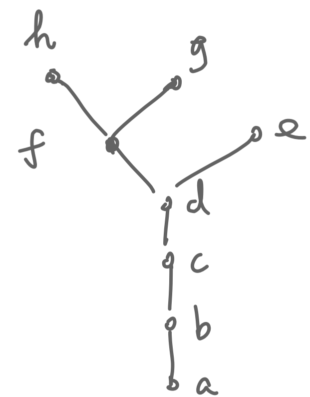
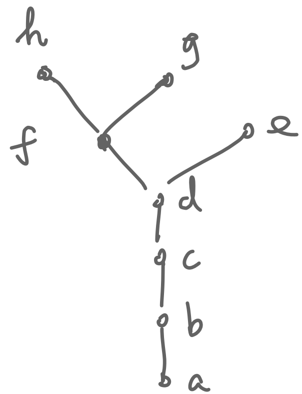


Problem: Prove that if a tree has n vertices, then it has $n-1$ edges.

S_n

e.g.



$|V| = 8$

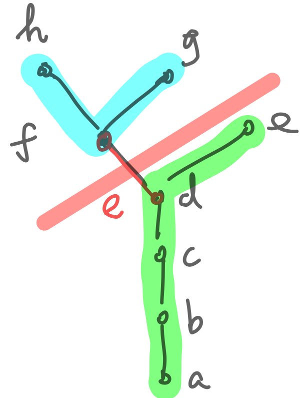
$|E| = 7$

Sketch of pf by strong induction. We prove the claim by strong induction on n.

Base case: Consider $n = 1$. A tree with one vertex has no edge ($\circ$), so S_1 holds.

Inductive step: Suppose S_n holds for n in the list $1, 2, 3, \ldots, k$ for some $k \geq 1$. We want to show that S_{k+1} holds, i.e., we want to show that a tree with $k+1$ vertices has k edges.

e.g.



Suppose $T=(V,E)$ is a tree with $k+1$ vertices.

Pick any edge $e \in E$ of T and remove it (but do not remove the two vertices it connects).

Claim: Doing so results in two smaller trees whose numbers of vertices add to $k+1$.

Suppose the number of vertices in the two parts are x and y.

Then $x, y > 0$ and $k+1 = x+y$. (so $x, y < k+1$)

By induction, the two smaller trees have $\underbrace{x-1 \text{ edges}}_{S_x}$ and $\underbrace{y-1 \text{ edges}}_{S_y}$. Thus, originally T has

$$\underset{x\text{-part}}{(x-1)} + \underset{y\text{-part}}{(y-1)} + \underset{e}{1} = x+y-2+1 = x+y-1 = k.$$

So S_{k+1} holds.

Now we are done. □