Math 2001. Lecture 3. HW1 posted online 01. 14. 2022.

Last time:

- Cartesian products of set and related counting problems
- Power sets of sets

Conjecture: $|\mathcal{P}(A)| = 2^{|A|}$

Today:

- proof of the conjecture
- set operations and Venn diagrams:

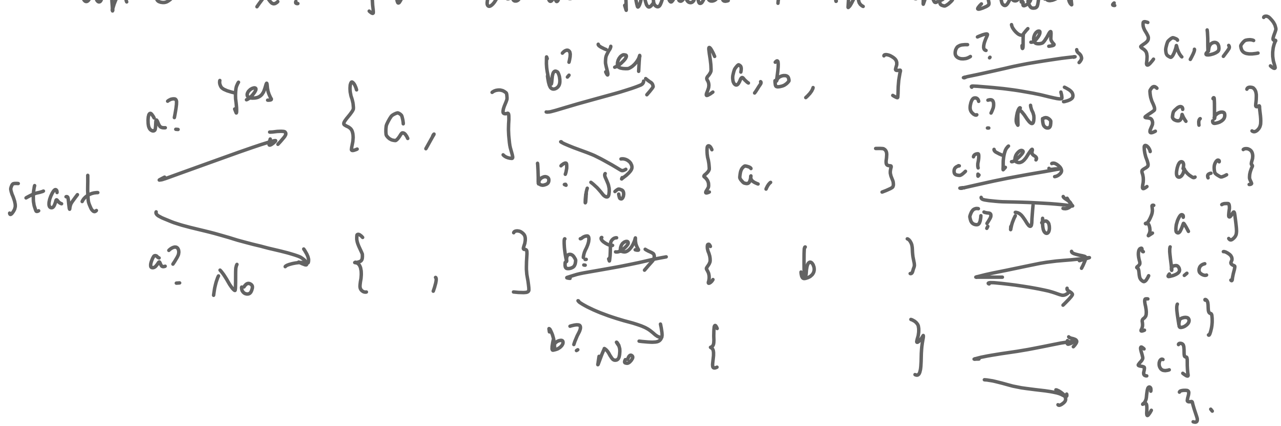
  union, intersection, difference, complement

1. $|\mathcal{P}(A)| = 2^{|A|}$

Prop: Let A be a finite set. Then $|\mathcal{P}(A)| = 2^{|A|}$.

Pf / Explanation: To specify a subset of A is to determine whether each elt of A should be included or not in the subset. We have 2 choices for each elt, hence there are $2 \times 2 \times \cdots \times 2 = 2^{|A|}$ choices for the subset. $\square$

E.g. $A = \{a, b, c\}$, $|A| = 3$. Consider constructing a subset of A.
Write "x?" for "do we include x in the subset?"



2. Set operations: new from old

We now introduce some operations to produce new sets out of old ones.

Def: Let A, B be two sets.

- The union of A and B is the set $A \cup B := \{x : x \in A \text{ or } x \in B\}$
- The intersection of A and B is the set $A \cap B := \{x : x \in A \text{ and } x \in B\}$
- The difference of A and B is the set $(A \setminus B =) A - B := \{x : x \in A \text{ and } x \notin B\}$

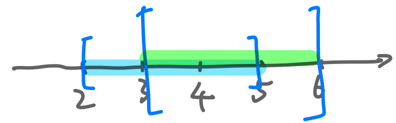
Eg. Let $A = \{1,2,3,4\}$, $B = \{3,4,5,6,7\}$, $C = \{2,6,7,9\}$.

Then

- $A \cup B = \{1,2,3,4,5,6,7\} = B \cup A$.
- $A \cap B = \{3,4\} = B \cap A$, $A \cap C = \{2\} = C \cap A$.
- $A - B = \{1,2\}$. $B - A = \{5,6,7\}$ → Note that $A - B \neq B - A$.
- $A - (B \cup C) = A - \{2,3,4,5,6,7,9\} = \{1\}$.

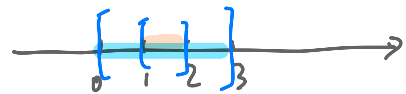
E.g. In the usual interval notation for $\mathbb{R}$. $\longrightarrow \mathbb{R}$.

$[2,5] \cup [3,6] = [2,6]$



$[2,5] \cap [3,6] = [3,5]$

$[0,3] - [1,2] = [0,1) \cup (2,3]$



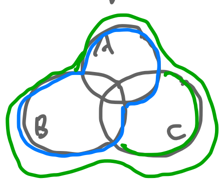
$[1,2] - [0,3] = \phi$.

Note: In general, $\cdot$ $A - B \neq B - A$. but $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

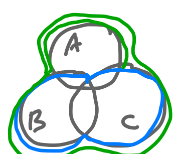
$\cdot$ $(A \cup B) \cup C = A \cup (B \cup C)$ since both sets consist of elts in at least one of A, B, and C.

So we can omit the parentheses

Similarly, $(A \cap B) \cap C = A \cap (B \cap C)$ (elts in all of A, B, C)



$A \cup B \cup C =$



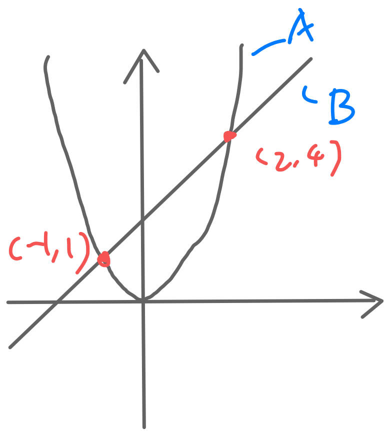
Ex. try to illustrate this with a (Venn) diagram.

Ex. Do we always have $A \cap (B \cup C) = (A \cap B) \cup C$?

Eg. Consider the sets $A = \{ (x, x^2) \mid x \in \mathbb{R} \} = \{ (x,y) \mid y = x^2 \} \subseteq \mathbb{R}^2$

and $B = \{ (x, x+2) \mid x \in \mathbb{R} \} = \{ (x,y) \mid y = x+2 \} \subseteq \mathbb{R}^2$.

Then $A \cap B = \left\{ (x,y) \,\middle|\, \begin{cases} y = x^2 \\ y = x+2 \end{cases} \right\} = \{ (-1, 1), (2,4) \}$.



3. Complement

Universe: when talking about sets of objects, we often have in mind a universal set or universe U of "all (relevant) objects".

(Sometimes we will need to specify U, but it's often clear from context.

Def: The complement of a set A (in a universe U) is the set $\bar{A} := \{x : x \notin A\}$ $(= \{x \in U : x \notin A\})$.

E.g.
- If $U = \mathbb{Z}$ and A is the set of all even integers, then $\bar{A}$ is the set of all odd integers.
- If $U = \{1,2,3,4,5,6\}$ and $A = \{2,3,5\}$, then $\bar{A} = \{1,4,6\}$.

Next time: more on set ops/Venn diagrams.