Math 2001. Lecture 29.

04. 06. 2022.

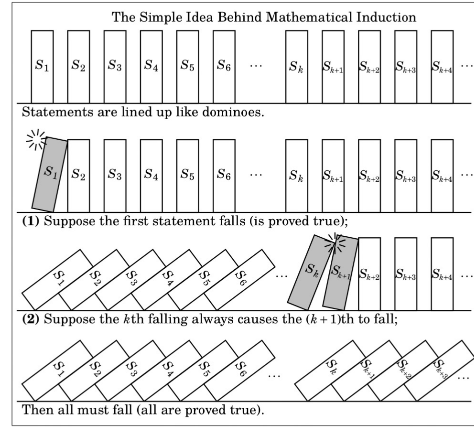
Last time:

- Mathematical induction

Steps:

- base case: proving S_1
- inductive step: proving $S_k \Rightarrow S_{k+1}$ $\forall k \geq 1$.

Key: to connect objects from S_k to objects in S_{k+1}.



Today:

- more examples of inductive proofs
- a variation of induction: strong induction.

1. More examples of mathematical induction

Prop: We have $2^n \overset{(*)}{\leq} 2^{n+1} - 2^{n-1} - 1 \quad \forall n \in \mathbb{Z}_{\geq 1}$.

e.g. $n=1$. Want $2^1 \leq 2^2 - 2^0 - 1$, i.e., $2 \leq 4-1-1$. ✓

$n=2$. want $2^2 \leq 2^3 - 2^1 - 1$, i.e., $4 \leq 8-2-1$. ✓

$n=3$. want $2^3 \leq 2^4 - 2^2 - 1$. i.e., $8 \leq 16-4-1$. ✓

Pf: We use induction on n.

Base Case: $n=1$. We have $2^n = 2^1 = 2$ and $2^{n+1} - 2^{n-1} - 1 = 2^2 - 2^0 - 1 = 4-1-1 = 2$.

so $2^n \leq 2^{n+1} - 2^{n-1} - 1$.

Inductive step: ($S_k \Rightarrow S_{k+1}$ $\forall k \geq 1$) Suppose $2^k \overset{①}{\leq} 2^{k+1} - 2^{k-1} - 1$ for some k.

Then $2^{k+1} = 2 \cdot 2^k \overset{①}{\leq} 2 \cdot (2^{k+1} - 2^{k-1} - 1) \underset{\text{algebra}}{=} 2^{k+2} - 2^k - 2 \underset{\text{algebra}}{<} 2^{k+2} - 2^k - 1$.

So $2^{k+1} \leq 2^{(k+1)+1} - 2^{(k+1)-1} - 1$.

It follows that $(*)$ holds for all $n \in \mathbb{Z}_{\geq 1}$.

Prop: If $n \in \mathbb{Z}_{\geq 1}$. we have $\underbrace{(1+x)^n \overset{(*)}{\geq} 1+nx \text{ for all } x \in \mathbb{R} \text{ with } x > -1}_{S_n}$.

Eg. $n=1$: want $(1+x)^1 \geq 1+1\cdot x \ \forall x > -1$. i.e., $1+x \geq 1+x \ \forall x > -1$. ✓

$n=2$: want $(1+x)^2 \geq 1+2\cdot x \ \forall x > -1$. i.e., $1+2x+x^2 \geq 1+2x \ \forall x > -1$.

true since $x^2 \geq 0$.

$n=3$: want $(1+x)^3 \geq 1+3\cdot x \ \forall x > -1$, i.e., $1+3x+\underline{3x^2+x^3} \geq 1+3x \ \forall x > -1$.

Pf: We use induction on n.

<u>Base case:</u> $n=1$. We have $(1+x)^n = 1+x$ and $1+n\cdot x = 1+x$,

so certainly $(*)$ holds for all $x \in \mathbb{R}$, $x > -1$.

Inductive step: ($S_k \Rightarrow S_{k+1}$ $\forall k \geq 1$) Suppose S_k holds for some $k \geq 1$. i.e.,

Suppose we have $(1+x)^k \overset{(1)}{\geq} 1 + k \cdot x$ $\forall x \in \mathbb{R}, x > -1$.

We want to show $(1+x)^{k+1} \geq 1 + (k+1) \cdot x$ $\forall x \in \mathbb{R}, x > -1$.

To do so, we note that $\forall x > -1$,

$$(1+x)^{k+1} = (1+x)^k \cdot (1+x) \overset{(1)}{\geq} (1+kx) \cdot (1+x)$$ since $1+x>0$ now that $x>-1$.

$$= 1 + kx + x + kx^2$$

$$= 1 + (k+1)x + kx^2$$

$$= \left(1+(k+1)x\right) + kx^2$$

$$\geq 1+(k+1)x$$ since $k>0$ and $x^2 \geq 0$.

i.e., S_{k+1} holds.

It follows that S_n holds for all $n \in \mathbb{Z}_{\geq 1}$. $\square$

2. Strong mathematical induction

Goal: To prove that a statement S_n (depending on n) holds for all n.

Outline: 1. Base case(s): prove that S_1 is true, or that the first several S_n ($S_1, S_2, \dots S_k$) are true.

2. (Strong inductive step): prove that for any $k \geq 1$, we have

$$S_1 \underbrace{\wedge}_{\text{"and"}} S_2 \wedge \cdots \wedge S_k \Rightarrow S_{k+1}.$$

Difference from basic induction: in the inductive step, we now prove S_{k+1} using not just S_k but all of $S_1, S_2, \dots, S_k$.

However, the use of recursion (knocking down "the next domino" using the previous case(s)) is still the main idea.

Examples:

Prop: Any postage of 8 cents or more can be formed by combining 3-cent stamps and 5-cent stamps.

Analysis: "Baby / base cases":

suffice as the base cases!

$$\left\{\begin{array}{l} 8c \to 8 = 3+5 \\ 9c \to 9 = 3+3+3 \end{array}\right.$$

$10c \to 10 = 5+5$

$11c \to 11 = 3+3+5$

$12c \to 12 = 3+3+3+3$

$\boxed{\text{Next time: turn these ideas into a pf by strong induction.}}$

Idea: Once we know how to form x cents, we know how to form $(x+3)$ cents

In other words, to form y cents for y large enough, we can reduce the problem to one about $(y-3)$-cents

recursion.