Math 2001. Lecture 28.

Final Exam: Wed, May 4. 01:30-4:00 pm. 04. 04. 2022.

Last time: Midterm II.

Today: · Mathematical induction (Ch. 11)

1. What is mathematical induction?

- The (type of) problem: Prove a statement S_n (depends on n) for all $n \geq 0$.
  (or $n \geq 1$, $n \geq 2$, ...)

  E.g. Show that for every $n \in \mathbb{Z}_{\geq 0}$, we have

  $$\underbrace{1 + 2 + \cdots + n = \frac{n(n+1)}{2}}_{S_n}.$$

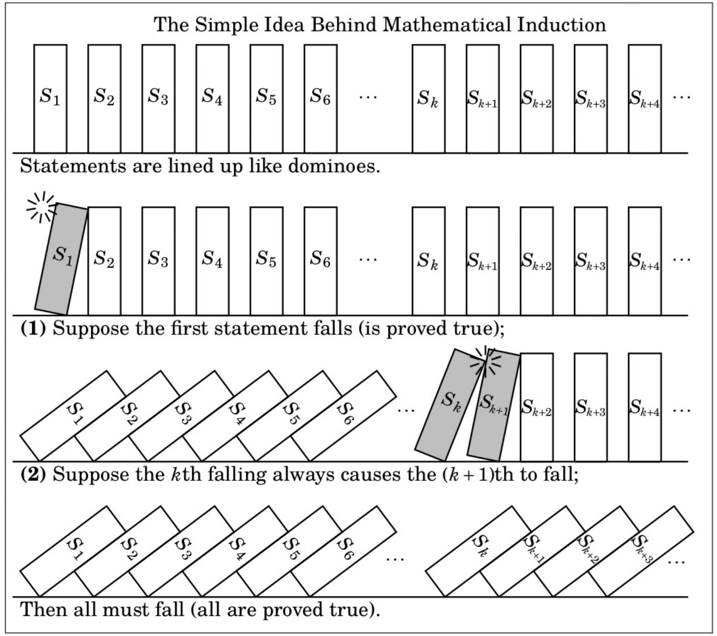
- The proof strategy: to prove "S_n is true for all n" in two steps:

  (1) the base case / basis step: prove S_n for the small possible n, i.e., prove the "first statement".

  (2) the inductive step: prove that for any k, if S_k holds then S_{k+1} holds.
  (S_k: the "inductive hypothesis")

  It then follows that S_n is true for all n.

- The intuitive idea:



- * The key in a typical proof by (mathematical) induction: to help prove the inductive step.

  study how the quantities/objects/expressions in S_k relate to those in S_{k+1}. (recursion)

2. Examples

Problem 1: $\forall n \in \mathbb{Z}_{\geq 1}$. $\quad \underbrace{1+2+3+\cdots+n = \frac{n(n+1)}{2}}_{S_n}$.

Pf: We use mathematical induction.

The base case: $n=1$. S_1 just says $1 = \frac{1(1+1)}{2}$, i.e., $1 = \frac{1\cdot 2}{2}$.

This is clearly true.

The inductive step: We need to show that $\underbrace{S_k \Rightarrow S_{k+1}}_{\text{if-then}}$ for each $k \in \mathbb{Z}_{\geq 1}$. So suppose S_k holds. i.e., suppose we know

(1) $\quad 1+2+3+\cdots+k = \frac{k(k+1)}{2}$. $\quad$ (known)

We want to show S_{k+1}, i.e.,

(2) $\quad 1+2+3+\cdots+k+(k+1) = \frac{(k+1)(k+1+1)}{2}$. $\quad$ (wanted)

(key: use the known to understand the wanted.)

To prove (2), note that

$$\underbrace{1+2+\cdots+k+(k+1)}_{\text{LHS}} = (1+2+\cdots+k) + (k+1)$$

$$\overset{(1)}{=} \frac{k(k+1)}{2} + (k+1)$$ hope this $\Rightarrow$ the RHS of (2)

$$= (k+1)\left(\frac{k}{2} + \frac{2}{2}\right)$$ algebraic manipulation

$$= (k+1)\frac{(k+2)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

It follows that $S_k \Rightarrow S_{k+1}$. So we are done with the inductive step.

It follows by mathematical induction that $1+2+\cdots+n = \frac{n(n+1)}{2}$ $\forall n \in \mathbb{Z}_{\geq 1}$. $\square$

Problem 2. If $n \in \mathbb{Z}_{>0}$, then $1+3+5+7+\cdots+(2n-1) \overset{\swarrow S_n}{=} n^2$.

e.g. $n=1$: $1=1^2$; $n=2$, $1+3=4=2^2$; $n=3$, $1+3+5=9=3^2$. ...

Pf: We use induction.

Base case: $n=1$. S_n says $1=1^2$, which is certainly true.

Inductive step. We need to prove $S_k \Rightarrow S_{k+1}$ $\forall k \in \mathbb{Z}_{>0}$. So let $k \in \mathbb{Z}_{>0}$ and assume S_k,

i.e., assume $1+3+5+\cdots+(2\underset{n=k}{k}-1) = k^2$ (1).

We need to prove $1+3+5+\cdots+(2k-1)+(2\underset{n=k+1}{(k+1)}-1) = (k+1)^2$ (2).

To prove (2), note that

$$1+3+\cdots+(2k-1)+(2(k+1)-1) = k^2+(2k+1) \overset{\text{algebra}}{=} (k+1)^2.$$

We have proved the inductive step.

It follows that $1+3+\cdots+(2n-1)=n^2$ $\forall n \in \mathbb{Z}_{>0}$. $\square$

Problem 3. If $n \in \mathbb{Z}_{\geq 0}$, then $\underbrace{5 \mid (n^5 - n)}_{S_n}$.

$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$

$1 \quad k$

e.g. $0^5 - 0 = 0$, $1^5 - 1 = 0$, $2^5 - 2 = 30$, $3^5 - 3 = 240$ are divisible by 5.

Pf: We use induction.

The base case: $n = 0$. $n^5 - n = 0 - 0 = 0$, so $5 \mid n^5 - n$ and the base case holds.

The inductive step: We suppose S_k holds for some $k \in \mathbb{Z}_{\geq 0}$ and deduce S_{k+1}.

Since S_k holds, we know $5 \mid k^5 - k$. We want to show that $5 \mid (k+1)^5 - (k+1)$

$$(k+1)^5 - (k+1) = 1 \cdot k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + \cancel{1} - k - \cancel{1}.$$

$$= (k^5 - k) + (5k^4 + 10k^3 + 10k^2 + 5k)$$

By the inductive hypothesis, $5 \mid (k^5 - k)$, and we note that $5 \mid 5k^4 + 10k^3 + 10k^2 + 5k$ because 5 divides each of $5k^4$, $10k^3$, $10k^2$, $5k$. Therefore $5 \mid (k+1)^5 - (k+1)$.

It follows by induction that $5 \mid n^5 - n$ $\forall n \in \mathbb{Z}_{\geq 0}$.

Problem 4. If $n \in \mathbb{Z}_{\geq 0}$, then $\underbrace{\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1}_{S_n}$.

E.g. $n=0$. " $0 \cdot 0! = 1! - 1$ "

$n=1$. " $0 \cdot 0! + 1 \cdot 1! = 2! - 1$ "

$n=2$ " $0 \cdot 0! + 1 \cdot 1! + 2 \cdot 2! = 3! - 1$ ".

$n=3$ " $0 \cdot 0! + 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! = 4! - 1$ "

Pf: We use math. induction on n.

Base case: $n=0$. we need $0 \cdot 0! = 1! - 1$, i.e. $0 \cdot 1 = 1 - 1$, which is true.

Inductive step: Suppose S_n holds when $n=k$ for some $k \in \mathbb{Z}_{\geq 0}$. we hope to show S_n holds when $n=k+1$. ...... (E.x.).