Math 2001. Lecture 26

Midterm 2: This Friday, April 1

03.28.2022.

Last time:

- example proofs from Ch.7.
- proving set containments ($A \subseteq B$, Ch. 8)

  To prove $A \subseteq B$, show that if $x \in A$, then $x \in B$.

Today:

- set equalities ($A = B$)
- disproofs (Ch. 9.)

1. Proving set equalities

To prove $A = B$ for sets A, B, we usually just prove that $A \subseteq B$ and $B \subseteq A$.

E.g. (8.10) Prove that $\{n \in \mathbb{Z} : 35 \mid n\} = \{n \in \mathbb{Z} : 5 \mid n\} \cap \{n \in \mathbb{Z} : 7 \mid n\}$.

pf: Let A, B be the LHS and RHS, respectively.

($A \subseteq B$) Suppose $n \in A$. Then $35 \mid n$. Since $5 \mid 35$, it follows that $5 \mid n$.
Since $7 \mid 35$, it follows similarly that $7 \mid n$.
So $n \in \{n \in \mathbb{Z} \mid 5 \mid n\}$ and $\{n \in \mathbb{Z} \mid 7 \mid n\}$, So $n \in B$.

($B \subseteq A$) Suppose $n \in B$. Then $5 \mid n$ and $7 \mid n$. It follows that in the unique prime decomposition of n, we must have $n = 5 \cdot 7 \cdot \underline{\text{(other primes)}}$, therefore $n = 35 \cdot \ell$ for some $\ell \in \mathbb{Z}$, so $35 \mid n$ and $n \in A$.

By the above, we have $A = B$. $\square$

(8.11.) Suppose A, B, C are sets, with $C \neq \emptyset$. Prove that if $A \times C = B \times C$, then $A = B$.

$$A \times C = \{(x,y) : x \in A, y \in C\}$$

Pf: Suppose $A \times C = B \times C$. We need to show that $A = B$.

$A \subseteq B$: Let $a \in A$. Take any elt $c \in C$, which can be done since $C \neq \emptyset$.

Then $(a,c) \in A \times C$. Since $A \times C \subseteq B \times C$, it follows that $(a,c) \in B \times C$.

Thus, $a \in B$ by the def. of $B \times C$.

So $A \subseteq B$.

$B \subseteq A$: A similar argument shows that $B \subseteq A$.

It follows that $A = B$.

8.13. Let A, B, C be sets. Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Ex: Do the $\subseteq$ & $\supseteq$ proof.

An alternative proof: Note that

$$\begin{aligned}
LHS &= \{(x,y) \mid x \in A \text{ and } y \in B \cap C\} \\
&= \{(x,y) \mid x \in A \text{ and } y \in B \text{ and } y \in C\} \\
&= \{(x,y) \mid (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C)\} \\
&= \{(x,y) \mid x \in A \text{ and } y \in B\} \cap \{(x,y) \mid x \in A \text{ and } y \in C\} \\
&= (A \times B) \cap (A \times C) = RHS.
\end{aligned}$$

$\square$

2. Disproofs

- To disprove a universal statement " $\forall s \in S, P(s)$ ", find a counter-example.

Eg. Statement : Every prime number is odd.

Proof that the statement is false : The number 2 is prime but not odd. $\square$

- To disprove a conditional statement " if P then Q ", find a counterexample for which P holds but Q fails.

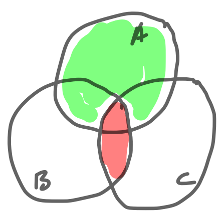
Eg. If $n \in \mathbb{Z}$, then $n^2 - n + 11$ is prime.

One disproof : Consider $n = 11 \in \mathbb{Z}$, in which case $n^2 - n + 11 = n^2 = 11^2$ is not prime.

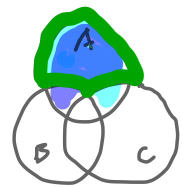
(9.2) Prove or disprove the following conjecture.

Conjecture: if A, B, C are sets, then $A-(B\cap C) = (A-B)\cap(A-C)$.

Analysis: should this be true?



LHS



RHS

Ex: Construct a counter example to disprove the conjecture.