Math 2001. Lecture 21.

03.07.2022.

Last time:

- Congruence of integers / modular arithmetic
- Contrapositive proofs: to prove "$P \Rightarrow Q$", show "$\sim Q \Rightarrow \sim P$" instead

Today:

- proof by contradiction:
  proving a conclusion by showing that the negation of it would lead to a problem, i.e., by assuming the negation and derive a contradiction.
- recommended practices for mathematical writing.

1. Proof by contradiction

$2.5 = \frac{10}{4} = \frac{5 \cdot 2}{2 \cdot 2} = \frac{5}{2}$

Examples:

Prop: The real number $\sqrt{2}$ is not rational, i.e., it can't be written in the form $\sqrt{2} = \frac{m}{n}$ for integers m, n where $n \neq 0$.

Pf: Suppose otherwise, i.e., suppose, for the sake of contradiction, that $\sqrt{2} = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$ where $n \neq 0$. Then we may pick m, n s.t. $\sqrt{2} = \frac{m}{n}$ and m, n share no integer factors other than 1 (i.e. they are coprime). Squaring both sides of the eq. gives $2 = \frac{m^2}{n^2}$, so $m^2 = 2n^2$. In particular, m^2 is even, which implies m is even, so we can write $m = 2k$ for some $k \in \mathbb{Z}$. Thus, we have $(2k)^2 = 2n^2$, so $4k^2 = 2n^2$, so $n^2 = 2k^2$, therefore n^2 is even, hence n is even. This implies that m, n are both divisible by 2, contradicting the assumption that m, n are coprime, so $\sqrt{2}$ is irrational. $\square$

Ex: Prove by contradiction that $\sqrt{3}$ is irrational.

Note: to prove a conditional statement ("$P \Rightarrow Q$") by its contrapositive is to suppose $\sim Q$ and prove $\neg P$, a contradiction to P. So contrapositive proofs are special cases of pfs by contradiction.

E.g. Prop: Let $a \in \mathbb{Z}$. If $\underbrace{a^2 \text{ is even}}_{P}$, then $\underbrace{a \text{ is even}}_{Q}$. ← actually we know and just used this.

Pf: Suppose a^2 is even. We want to prove a is even. $\underbrace{\text{Suppose otherwise}}_{\sim Q}$, i.e., $\underbrace{\text{Suppose } a \text{ is odd}}_{\sim Q}$. Then $a = 2k+1$ for some $k \in \mathbb{Z}$. So

$$a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

therefore $\underbrace{a^2 \text{ is odd}}_{\sim P}$. This contradicts the assumption that a^2 is even, so a must be even. $\square$

(We have proved the contrapositive of the desired claim, so we are done.)

2. Writing advices

1. Begin each sentence with a word, not a math symbol.

   "A is a set of B." ✗ $\longrightarrow$ "The set A is a subset of B." ✓

2. End each sentence with a period, even if the sentence ends with a symbol or expression.

   "The binomial thm states that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$" ✗

   "The binomial thm states that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$." ✓

3. Separate math symbols/expressions with words.

   "As $x^2-1=0$, $x=1$ or $x=-1$." ✗ $\longrightarrow$ "As $x^2-1=0$, we have $x=1$ or $x=-1$."

4. Avoid misuse of symbols.

e.g. $1 \in \mathbb{Z}$, ✓ "$1 \subseteq \mathbb{Z}$" ✗ $\{1\} \subseteq \mathbb{Z}$ ✓

5. Avoid unnecessary symbols.

No set ~~X~~ has negative cardinality.

6. Use the first person plural.

"We" rather than "I", as in "We have / we conclude".

7. Use the active voice.

8. Introduce / declare / describe/quantify new symbols.

"Since $a|b$, we have $b = ac$." ✗

"Since $a|b$, we have $b = ac$ for some $c \in \mathbb{Z}$." ✓

9. Avoid "it" / "this", etc., if there might be ambiguity.

10/11. Use conjunctions in suitable places.

(since, because, hence, therefore, ...)

12. Be clear!