Math 2001. Lecture 2. 01. 12. 2022.

Last time:

- notation for sets: set builder notation $\{ \text{expression} \mid \text{rule} \}$

  the empty set is denoted by ϕ.

- Subsets: "$B \subseteq A$" if every elt of B is an elt of A.

  Note: It is vacuously true that for any set A, every elt in ϕ is in an elt of A, so $\phi \subseteq A$.

- two kinds of containment: $\begin{cases} \text{"}a \in A\text{": } a \text{ is an elt of } A. \\ \text{"}B \subseteq A\text{": } B \text{ is a subset of } A. \end{cases}$

- An unfinished proof: $A = \mathbb{Z}$ for $A = \{ 2a+5b \mid a, b \in \mathbb{Z} \}$

Today:

- finishing the pf
- Cartesian products of sets and a counting principle
- power sets of sets

1. The proof that $A = \mathbb{Z}$ for $A = \{ 2a+5b \mid a, b \in \mathbb{Z} \}$

pf: We already noted that it suffices to show $A \subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq A$.

We also proved that $A \subseteq \mathbb{Z}$ because every elt of A is an integer.

So it remains to show that $\mathbb{Z} \subseteq A$.

"0 is in $\mathbb{Z}$, is it in A?" $\quad 0 = 2 \cdot 5 + 5 \cdot (-2) \in A$. ✓

"$1 \in \mathbb{Z}$, is it in A"? $\quad 1 = 2 \cdot (-2) + 5 \cdot 1 \in A$.

3 ? $\quad 3 = 1 \cdot 3 = 2 \cdot (-2 \cdot 3) + 5 \cdot (1 \cdot 3) \in A$

($\mathbb{Z} \subseteq A$): We first note that $1 = 2 \cdot (-2) + 5 \cdot 1$.

Thus, for any elt $k \in \mathbb{Z}$, $\quad k = 1 \cdot k = 2 \cdot (-2k) + 5 \cdot k$ where $-2k, k \in \mathbb{Z}$,

so $k \in A$.

It follows that $\mathbb{Z} \subseteq A$, and we are done.

2. Cartesian products of sets

Def: The Cartesian product of two sets A and B is the set

$$A \times B := \{ (a,b) : a \in A, b \in B \}.$$

More generally, the Cartesian product of a sequence of sets $A_1, A_2, \dots, A_k$ is the set

$$A_1 \times A_2 \times \cdots \times A_k = \{ (a_1, a_2, \dots, a_k) : a_i \in A_i \ \forall 1 \le i \le k \}$$

Eg. (Menu example!) Let $A = \{\text{burger}, \text{pizza}, \text{hotdog}\}$, $B = \{\text{Coke}, \text{Sprite}\}$.

Then $A \times B = \left\{ \begin{array}{lll} (\text{burger}, \text{Coke}), & (\text{pizza}, \text{Coke}), & (\text{hotdog}, \text{Coke}) \\ (\text{burger}, \text{Sprite}), & (\text{pizza}, \text{Sprite}), & (\text{hotdog}, \text{Sprite}) \end{array} \right\}.$

Note $|A \times B| = |A| \times |B| = 3 \cdot 2 = 6$

In particular, if A and B describe the food and drink options at a restaurant, then $A \times B$ describes all the food-drink combo options.

E.g. (Dice Example) Think of the set $S = \{1, 2, 3, 4, 5, 6\}$ as the set of outcomes when you throw and read a dice. then

$$S \times S = \left\{ \begin{matrix} (1,1), & (1,2), & \cdots & \\ (2,1), & & \cdots & \\ \vdots & & & \\ (6,1), & \cdots & \cdots & ,(6,6) \end{matrix} \right\}$$

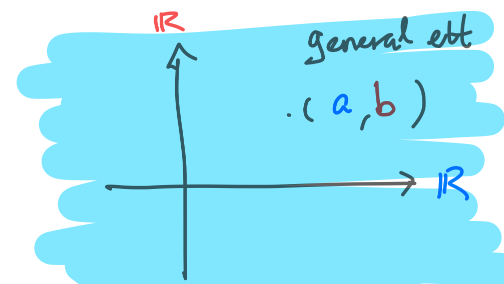
conveniently encodes the outcomes for throwing and reading the dice twice in a row.

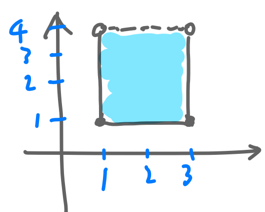
Note that $|S \times S| = |S| \times |S| = 6 \times 6 = 36$.

"x" "y" → usual interval notation

E.g. $\mathbb{R}^2$ and $[1,3] \times [1,4) \subseteq \mathbb{R}^2$.

Def: $\mathbb{R}^2$ stands for $\mathbb{R} \times \mathbb{R}$, visualized as





Note: From the menu and dice examples, we note the following

think of the <u>dynamical</u> process of "building" $A \times B$.

Prop: We have $|A \times B| = |A| \cdot |B|$ for any sets A and B.

More generally, $|A_1 \times A_2 \times \cdots \times A_k| \overset{*}{=} |A_1| \cdot |A_2| \cdots |A_k|$ for any sets $A_1, A_2, \ldots, A_k$.

In particular, the product $A_1 \times \cdots \times A_k$ is finite if and only if $A_1, \ldots, A_k$ are all finite.

Rmk: Taking the Cartesian product of a set A with itself gives Cartesian powers

$$A^k = \underbrace{A \times A \times \cdots \times A}_{k \text{ times}}$$

3. Power set of sets

Def: The <u>power set</u> of a set A is the set of all subsets of A.

We denote the power set by $\mathcal{P}(A)$.

E.g. $A = \{1, 2\} \implies \mathcal{P}(A) = \{\ \phi, \{1\}, \{2\}, \{1, 2\}\ \}$

Q: What is $|\mathcal{P}(A)|$ in general (in terms of A) ?

Meta Q : How do we approach this question ?

- start with small (baby) examples !

Baby cases:

- $|A| = 0$, i.e. $A = \phi$. $\Rightarrow$ $P(A) = \{\phi\}$. $\Rightarrow$ $|P(A)| = 1$
- $|A| = 1$, say $A = \{a\}$ $\Rightarrow$ $P(A) = \{\phi, \{a\}\}$ $\Rightarrow$ $|P(A)| = 2$
- $|A| = 2$, say $A = \{a, b\}$ $\Rightarrow$ $P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$ $\Rightarrow$ $|P(A)| = 4$
- $|A| = 3$, say $A = \{a, b, c\}$ $\Rightarrow$ $P(A) = \left\{ \begin{array}{c} \phi, \{a\}, \{b\}, \{c\} \\ \{a,b\}, \{a,c\}\ \{b,c\} \\ \{a,b,c\}. \end{array} \right\}$ $\Rightarrow$ $|P(A)| = 8$.
- $|A| = 4$ $\xrightarrow{\text{Ex.}}$ $\Rightarrow |P(A)| = 16$.

Conjecture: $|P(A)| = 2^{|A|}$.

Q: Can you prove this by "dynamic counting"?