Math 2001. Lecture 19.

03.02.2022.

Last time: · direct proofs of conditional statements (with template)

Today: · more examples · practice proofs

1. Examples

(a) Prove that if $x, y \in \mathbb{Z}$ and x is even, then xy is even.

Pf: Suppose $x, y \in \mathbb{Z}$ and x is even.

Then $x = 2k$ for some $k \in \mathbb{Z}$.

Therefore $xy = 2ky = 2(ky)$.

(ie, $xy = 2N$ for some integer N, namely, for $N = ky$.)

So xy is even, so we are done.

(b) Show that if $n \in \mathbb{Z}$, then $n^2 + 3n + 4$ is even.

Pf: Suppose/Let $n \in \mathbb{Z}$. We consider two cases.

(1) n is even. Then $n = 2k$ for some integer k.

So $n^2 + 3n + 4 = (2k)^2 + 3 \cdot 2k + 4 = 4k^2 + 6k + 4 = 2(2k^2 + 3k + 2)$

where $(2k^2 + 3k + 2) \in \mathbb{Z}$ since $k \in \mathbb{Z}$.

So $n^2 + 3n + 4$ is even.

(2) n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$.

So $n^2 + 3n + 4 = (2k+1)^2 + 3(2k+1) + 4 = 4k^2 + 4k + 1 + 6k + 3 + 4$

$= 4k^2 + 10k + 8 = 2(2k^2 + 5k + 4)$

where $(2k^2 + 5k + 4) \in \mathbb{Z}$ since $k \in \mathbb{Z}$.

So $n^2 + 3n + 4$ is even.

These cases are exhaustive, so $n^2 + 3n + 4$ is even for every $n \in \mathbb{Z}$. $\square$

(6) Prove that every (integer) multiple of 4 equals $\overbrace{1+(-1)^n(2n-1)}^{x}$ for some nonnegative integer n.

$$\begin{cases} 2n-1+1 = 2n & \text{if } 2 \mid n \\ -(2n-1)+1 = -2n+2 & \text{if } 2 \nmid n \end{cases} \approx \pm 2n$$

"if an int. N is divisible by 4, then $N = 1+(-1)^n(2n-1)$ for some $n \in \mathbb{Z}_{\geq 0}$"

We prove the converse last time.

Trials: if $N=0$, what n works? $n=0$? ✓

if $N=4$, ... ? want $\pm 2n \approx 4$. how about $n=2$? ✓

if $N=8$, ... $\pm 2n \approx 8$, how about $n=4$? ✓

For any $N \geq 0$ $(4 \mid N)$, taking $n = N/2$ works.

if $N=-4$, ... ? want $-2n \approx -4$. how about $n=2$? ✗

how about $n=1$? $n=1 \mapsto x=0$ ✗

if $N=-8$. → want $-2n \approx -8$. 4, 3, 5 ✗ ✗ ✓ how about $n=3$? $n=3 \mapsto x=-4$ ✓

take $n = \frac{N}{-2}+1$

$x = N$. ✓

Pf: Suppose N is a multiple of 4. Then $N = 4k = 2(2k)$ for some $k \in \mathbb{Z}$.

So $N/2$ is an integer, as is $N/-2$. We want to find an int $n \in \mathbb{Z}_{\geq 0}$

s.t. $N = +1 + (-1)^n (2n-1)$. We discuss two cases.

(1) $N \geq 0$. Consider $n = N/2$. Then $N/2 = 2k \in \mathbb{Z}$ as mentioned. Moreover,

$1 + (-1)^n (2n-1) = 1 + (-1)^{2k} (2n-1) = (+1 \cdot (2n-1) = 2n = 2\frac{N}{2} = N$, as desired.

(2) $N < 0$. Consider $n = -\frac{1}{2}N + 1 = -2k + 1$, which is an odd int. Moreover,

$1 + (-1)^n (2n-1) = 1 - (2n-1) = -2n + 2 = (-2)(-\frac{1}{2}N + 1) + 2 = (N-2) + 2 = N$, as desired.

So $N = 1 + (-1)^n (2n-1)$ for some nonnegative int. n whenever $4 | n$. $\square$

2. Practice problems

See the problems on the proofs worksheet. The problems will also be HW problems.

Next time: Ch. 5. Contrapositive proof.