Midterm 1 this Friday.

Math 2135. Lecture 16.

See website for the 2nd worksheet on counting and a list of topics.

02.21.2022.

Last time: · Summary of four types of counting problems

- the pigeonhole principles: Place n objects into k boxes.
  - if $n > k$, then at least one box gets more than one objects.
  - if $n < k$, then at least one box gets zero object.
- the division principle: Place n objects into k boxes.

Correction/improvement: Last time I stateted that

· at least one box gets $\lfloor \frac{n}{k} \rfloor$ or more objects.

In fact, we have

Prop 1. at least one box gets $\lceil \frac{n}{k} \rceil$ or more objects.

e.g. Place 9 balls into 4 boxes. Then at least one box has $\lceil \frac{9}{4} \rceil = 3$ or more balls

Today: · more on the division principle · more on combinatorial proofs.

1. The division principle

Pf of Prop 1: (Place n objects into k boxes). Suppose not, ie., suppose that every one of the k boxes has less than $\lceil \frac{n}{k} \rceil$ objects.

$\lceil r \rceil - 1 < r$ $\forall r \in \mathbb{R}$

Now, · if $\frac{n}{k}$ is an integer, then each box has at most $\lceil \frac{n}{k} \rceil - 1 = \frac{n}{k} - 1$ objects,

so in total the boxes contain at most $k \cdot \left(\frac{n}{k} - 1\right) = n - k < n$ objects, contradiction.

· if $\frac{n}{k}$ is not an integer, then each box has at most $\lceil \frac{n}{k} \rceil - 1 = \lfloor \frac{n}{k} \rfloor$ objects.

So in total the boxes contain at most $k \cdot \lfloor \frac{n}{k} \rfloor < k \cdot \left(\frac{n}{k}\right) = n$ objects, contradiction.

It follows that some box has $\lceil \frac{n}{k} \rceil$ or more objects, as desired. $\square$

Statement of the "other half" of the division principle: Place n objects into k boxes. Then at least one box gets $\lfloor \frac{n}{k} \rfloor$ or fewer objects.

EX: Prove this other half.

Examples:

3.25. gumball machine: many red, green, blue and white gumballs.

Each gumball costs 5 cents.

Deal: buy some gumball, and if 13 of them have the same color you get \$5.

What is the fewest number of gumballs you need to buy to ensure that you will make money on the deal?

"Worst case scenario": You buy gumballs and get to the point where you have 12 gumballs of each color $\longrightarrow$ buying 48 gumballs

So, what if we buy 49 gumballs?

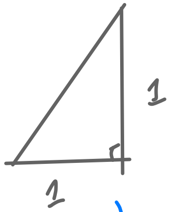
Soln: (a) It is not sufficient to buy 48 gumballs, since we might end up with 12 of each color in that case.

(b) On the other hand, if we buy 49 gumballs, then by the division principle some color will correspond to $\lceil \frac{49}{4} \rceil = \lceil 12.25 \rceil = 13$ or more gumballs, i.e., at least 13 of the 49 gumballs will have the same color.

By (a) and (b), 49 is the least number of gumballs we need to buy.

$$(500 - 5 \times 49 = 500 - 245 = \underbrace{255}_{\text{the money you make}} \text{ cents})$$

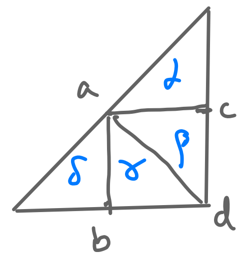
3.26.



Nine points are randomly placed in the △ on the left. Show that at least three points of the nine form a triangular whose area is 1/8 or less.

Area of the △ is $\frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$

Soln:



Let a, b, c be the midpoint of the three sides as shown. Then the regions α, β, γ, δ all have area $\frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$.

By the division principle, at least one region will end up getting $\left\lceil \frac{9}{4} \right\rceil = 3$ of the nine points. These three point form a △ inside that region, and this △ must have area $\frac{1}{8}$ or less, as desired. □

2. More on combinatorial identities (§3.10, seen already, will not be on the midterm)

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Example: $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

Pf: Consider taking n objects from the two sets $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$.

$2n$ diff. objects in total.

One approach: Form the union $A \cup B$, which has $2n$ objects and take n objects

$\rightarrow \binom{2n}{n}$ way to do this.

Another approach: Take k object from A and then $(n-k)$ objects from B, where $0 \leq k \leq n$.

$$\rightarrow \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2 \text{ ways to do so.}$$

It follows that $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

Next time:
- more comb. proofs
- review for midterm