

Last time: · more counting problems involving multisets

— int. solns to $0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq k$ — the word problem.

(spellings of [B, A, N, A, N, A])

Today: · Summary of types of counting problems

$$\frac{6!}{2! 3!}$$

· The pigeonhole and division principle.

1. Summary of counting problems

Let X be a set $\{x_1, \dots, x_n\}$ or a multiset $[x_1, x_2, \dots, x_n]$ of n elts.

We have learned how to count the following kinds of objects:

(1) **Permutation of a set**: X is a set, we take k elts out of X and line them up $\rightarrow \#(\text{Such line-ups/ permutations}) = P(n, k) = n(n-1)\dots(n-k+1)$
 \downarrow In particular, if $k=n$, then the above number is $P(n, n) = n!$
rep. not allowed, order matters.

(2) **Combination / Subset**: X is a set, we take k elts ($k \leq n$) out of X and form a subset / combination. $\rightarrow \#(\text{Such subsets}) = C(n, k) = \frac{n!}{k!(n-k)!}$
 \downarrow rep. not allowed, order doesn't matter.

(3) **Permutation of multiset (the word problem)**: X is a multiset whose elts have multiplicity p_1, p_2, \dots, p_k . We line up all the elts of X . $\rightarrow \#(\text{lineups/ permutations}) = \frac{n!}{p_1! p_2! \dots p_k!}$
 \downarrow rep allowed, order does matter; analog of the special case $k=n$ from (1).

(4) **"Multiset Combination"** (bars-and-stars method): we want to form a multiset of size k using elts from a set Y of size n . $\rightarrow \#(\text{Such multisets}) = \binom{k+n-1}{k}$.
 \downarrow rep allowed, order doesn't matter

2. The Pigeonhole Principle

The principle: Place n objects into k boxes where n, k are positive integers.

think 4 apples in 3 boxes
← (1) If $n > k$, then at least one box contains more than 1 object.

(2) If $n < k$, then at least one box contains no object.

Why? (1) Otherwise the boxes contain at most $k < n$ objects, a contradiction.

2 apples in 3 boxes
(2) Otherwise the boxes have at least $k > n$ objects, a contradiction.

Examples: 2-9-1. Show that if six integers are chosen at random, then

at least two of them have the same remainder when divided by 5.

boxes \leftrightarrow remainders: when dividing an integer by 5 there are only 5 possible remainders, namely, 0, 1, 2, 3, 4. Since $6 > 5$, some remainder must appear at least more than once when we divide the 6 numbers by 5. i.e., at least two of the six

Numbers have the same remainder, as claimed.

3.24. Pick six integers from $X = \{0, 1, 2, 3, \dots, 7, 8, 9\}$. Show that two of them must add to 9.

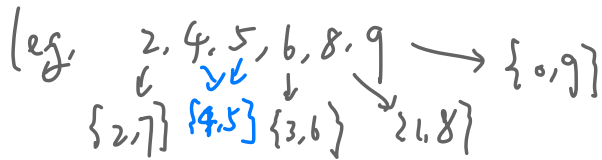
e.g. $\{2, 4, 5, 6, 8, 9\} \rightarrow 4+5=9.$

$$\{1, 3, 5, 6, 7, 8\} \rightarrow 3+6=9 = 1+8$$

Soln: Consider the five pairs ("boxes") $\{0, 9\}, \{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}$,

each of which consists of two numbers adding to 9. Now take any six

integers from X and place each of them in the pair containing it. By the



Pigeonhole principle, some pair $\{i, j\}$ contain two of these six numbers. These two numbers add to 9 by the design of the boxes. \square

3. The Division Principle.

→ We'll usually only consider positive numbers

Floors and ceilings: For a real number r ,

the floor of r , denoted by $\lfloor r \rfloor$, is the largest int. k with $k \leq r$.

e.g. $\lfloor 2 \rfloor = 2$, $\lfloor 2.1 \rfloor = 2$, $\lfloor \pi \rfloor = 3$

the ceiling of r , denoted by $\lceil r \rceil$, is the smallest int. k with $k \geq r$.

$\lceil 2 \rceil = 2$, $\lceil 2.1 \rceil = 3$, $\lceil \pi \rceil = 4$.

The principle: place n objects into k boxes.

• At least one box gets at least $\left\lceil \frac{n}{k} \right\rceil$ objects. average

e.g. 12 objects into 3 boxes. $\left\lceil \frac{12}{3} \right\rceil = 4 \rightarrow$ some box has at least 4 objects.

13 objects into 3 boxes. $\left\lceil \frac{13}{4} \right\rceil = 5 \rightarrow$ some box has at least 5 objects.

• **Ex:** formulate the analogous claim for upper bound. Next time: more on the principle.