Math 2001. Lecture 15.

02.18.2022

Last time: · more counting problems involving multisets

— int. sol'ns to $0 \le x_1 \le x_2 \le \cdots \le x_n \le k$ — the word problem.

$\#(\text{spellings of } [B, A, N, A, N, A]) = \dfrac{6!}{2!\,3!}$

Today: · Summary of types of counting problems

· The pigeonhole and division principle.

1. Summary of counting problems

Let X be a set $\{x_1, \dots, x_n\}$ or a multiset $[x_1, x_2, \cdots, x_n]$ of n elts.

We have learned how to count the following kinds of objects:

(1) Permutation of a set : X is a set, we take $k (\le n)$ elts out of X and line them up $\longrightarrow$ # (such line-ups / permutations) $= P(n,k) = n(n-1)\cdots(n-k+1)$

In particular, if $k=n$, then the above number is $P(n,n) = n!$

rep. not allowed, order matters.

(2) Combination / Subset : X is a set, we take k elts ($k \le n$) out of X and form a subset / combination. $\longrightarrow$ # (such subsets) $= C(n,k) = \frac{n!}{k!(n-k)!}$

rep. not allowed, order doesn't matter.

(3) Permutation of multiset (the word problem): X is a multiset whose elts have multiplicity $p_1, p_2, \cdots, p_k$. We line up all the elts of X. $\to$ # line-ups / permutations $= \frac{n!}{p_1! p_2! \cdots p_k!}$

rep allowed, order does matter; analog of the special case $k=n$ from (1).

(4) "Multiset Combination" (bars-and-stars method): we want to form a multiset of size k using elts from a set Y of size n. $\longrightarrow$ # (such multisets) $= \binom{k+(n-1)}{k}$.

rep allowed, order doesn't matter

2. The Pigeonhole Principle

The principle: Place n objects into k boxes where n, k are positive integers.

think 4 apples in 3 boxes ← (1) If $n > k$, then at least one box contains more than 1 object.

(2) If $n < k$, then at least one box contains no object.

Why? (1) Otherwise the boxes contain at most $k < n$ objects, a contradiction.

2 apples in 3 boxes (2) Otherwise the boxes have at least $k > n$ objects, a contradiction.

Examples: 3.9.1. Show that if six integers are chosen at random, then at least two of them have the same remainder when divided by 5.

boxes ⇔ remainders: When dividing an integer by 5 there are only 5 possible remainders, namely, 0, 1, 2, 3, 4. Since $6 > 5$, some remainder must appear at least more than once when we divide the 6 numbers by 5. i.e., at least two of the six numbers have the same remainder, as claimed.

3.24. Pick six integers from $X = \{0, 1, 2, 3, \cdots, 7, 8, 9\}$. Show that two of them must add to 9.

e.g. $\{2, 4, 5, 6, 8, 9\} \to 4+5=9$.

$\{1, 3, 5, 6, 7, 8\} \to 3+6 = 9 = 1+8$

Soln: Consider the five pairs ("boxes") $\{0, 9\}, \{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}$, each of which consists of two numbers adding to 9. Now take any six integers from X and place each of them in the pair containing it. By the Pigeonhole principle, some pair $\{i, j\}$ contain two of these six number. These two numbers add to 9 by the design of the boxes. $\square$

(e.g. $2, 4, 5, 6, 8, 9 \longrightarrow \{0, 9\}$

$\{2,7\}$ $\{4,5\}$ $\{3,6\}$ $\{1,8\}$

3. The Division Principle.

we'll usually only consider positive numbers

Floors and ceilings: For a real number r,

the floor of r, denoted by $\lfloor r \rfloor$, is the largest int. k with $k \le r$.

e.g. $\lfloor 2 \rfloor = 2$, $\lfloor 2.1 \rfloor = 2$, $\lfloor \pi \rfloor = 3$

the ceiling of r, denoted by $\lceil r \rceil$, is the smallest int. k with $k \ge r$.

$\lceil 2 \rceil = 2$, $\lceil 2.1 \rceil = 3$, $\lceil \pi \rceil = 4$.

The principle: place n objects into k boxes.

- At least one box gets at least $\left\lceil \frac{n}{k} \right\rceil$ objects.

  average

  e.g. 12 objects into 3 boxes. $\left\lceil \frac{12}{3} \right\rceil = 4 \to$ some box has at least 4 objects.

  13 objects into 3 boxes, $\left\lceil \frac{13}{4} \right\rceil = 5 \to$ some box has at least 5 objects

- Ex: formulate the analogous claim for upper bound.

Next time: more on the principle.