Math 2001. Lecture 14. Midterm 1 (Ch1-3): Friday, Feb 25. 02.16.2022.

Last time:
- multisets
- multisets formed using elts from a set, the bars-and-stars method

Today:
- more problems
- another problem related to multiset: the word problem
- Summary of types of counting problems

1. Examples for bars-and-stars

Warmup: · How many multisets of size 4 can be made from $\{a, b, c, d, e, f\}$?

(e.g. $* | * | * * | | | \leftrightarrow a\,b\,c\,c$) Answer: $\binom{4+(6-1)}{4}$

· 3.8.4 We have 20 red balls, 20 blue balls, 20 green, 20 white.
How many sets of 15 balls can we form out of these 80 balls?

Answer: $\binom{15+4-1}{4-1}$

· 3.8.6. 20 red, 20 blue, 20 green, 1 white, 1 black.
How many sets of 20 balls can we form?

difference: in the second problem the number of times white and black can be used is limited.

In light of the limit on # of white and black balls, we use the addition principle and discuss cases:

no black, no white: $\binom{20+3-1}{3-1}$; use the black but no white: $\binom{19+3-1}{3-1}$, ...

specify the other 19

Another warm-up: $\left(\#\text{ nonnegative int. solns to } w+x+y+z=10\right) = \binom{10+4-1}{4-1}$

Ex: "draw" the solns of $0 \le x \le y \le 3$.

A different problem:

3.21. What's $\left(\#\text{ int. tuples } (w,x,y,z) \text{ s.t. } 0 \le w \le x \le y \le z \le 10\right)$?

Note that such tuples are in bijection with stars-and-bars configurations with 10 stars and 4 bars via the encoding

(for the previous problem we had 3 bars!)

$$***|***||***|* \longrightarrow 3,6,6,9$$

stars-and-bars-graph $\rightarrow$ ($w=$(# * before the 1st bar), $x=$ #(*'s before the 2nd bar), $y=\ldots$, $z=\ldots$)

So: the desired number is $\binom{10+4}{4}$.

2. The word problem (permutations of multisets)

Answer: $\frac{4!}{2!} = 12$

Q: Using (all) the letters of MISSISSIPPI, how many different words/strings can we get?

Let's try an easier problem first: same question, but with BOOK.

Key idea: distinguish the multiple occurrences of each letter first (and then compensate)

B, O_1, O_2, K

perm. of $\{B, O_1, O_2, K\}$ → $4!$

$$\left\{ \begin{array}{c|c|c|c|c} B K O_1 O_2 & B O_1 K O_2 & B O_1 O_2 K & \cdots & O_1 O_2 B K \\ B K O_2 O_1 & B O_2 K O_1 & B O_2 O_1 K & & O_2 O_1 B K \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ BKOO & BOKO & BOOK & & OOBK. \end{array} \right.$$

overcounting fact: $2!$ (perm. of $\{O_1, O_2\}$)

Another example : BANANA.

If we distinguished the A's the strings formed out of $\{B, A_1, N_1, A_2, N_2, A_3\}$ are just permutations of these 6 letters, of which there are $6!$

But each string with such distinction corresponds to $\underbrace{3! \cdot 2!}$ strings formed from $[B, A, N, A, N, A]$. So

$$\text{BANANA} \to \begin{cases} B A_1 \underline{N_1} A_2 \underline{N_2} A_3 \\ B A_1 \underline{N_2} A_2 \underline{N_1} A_3 \\ B A_1 N_1 A_3 N_2 A_2 \\ B A_1 N_2 A_3 N_2 A_2 \\ \vdots \\ B A_3 N_2 A_2 N_1 A_1 \end{cases}$$

the # spellings of the letters of BANANA $= \dfrac{6!}{3!\ 2!}$

Point: If A is a multiset with n elts, where the elts have multiplicities $p_1, p_2, \dots, p_k$. Then the number of permutations of A (spellings using all the elts of A) is

$$\frac{n!}{p_1!\ p_2! \cdots p_k!}.$$

MISSISSIPPI : $\dfrac{11!}{1!\,2!\,4!\,4!}$

Next time:

Summary of counting problem types

3. A summary We have dealt with counting problems of several types:

(1) Permutation of a set: Consider a set X with n elts

(2) Combination / Subset:

(3) Permutation of multiset:

(4) "Multiset Combination":