Math 2001. Lecture 13.

02.14.2022.

Last time: Worksheet on permutations and combinations.

Today: A problem about multiset combinations, the bars and stars method.

Here, a multiset is a variation of a set where each elt can appear multiple times. The number of times an elt appears is called the multiplicity of the elt. We enclose multisets in [.].

e.g. as sets, $\{1,2,2,3,3,3\} = \{1,2,3\} = \{3,1,2\}$

as multisets, $[1,2,2,3,3,3] = [3,1,2,3,2,3]$ ("order still doesn't matter")

but $[1,2,3,3,3] \neq [1,2,2,3,3,3]$

1. Bars and stars (multiset combinations)

The question: Given a set X of size n, how many multisets of size k are there whose elts are selected from X?

E.g.: How many size-3 multisets can be made from the set $X = \{a, b, c, d\}$?

The numbers are small, so let's try to find all the sets: (write $\{x, y, z\}$ as xyz)

1 letter: aaa, bbb, ccc, ddd.

2 letters: aab, abb; aac, acc; aad, add; bbc, bcc; bbd, bdd; ccd, cdd.

3 letters. abc, abd, acd, bcd.

Total: $4 + 12 + 4 = 20$.

A better organization/encoding scheme: use bars to separate the different letters (so that a's appear before the first bar, b between the first and second bar, ...)

b c d

aaa| | | |bbb, |ccc, |ddd.

aa|b, a|bb ; aa|c, a|cc ; aa|d, a|dd ; |bbc, |bcc, |bbd, |bdd,

|ccd, cdd.

a|bc, a|bd, a|cd, bcd.

↓ |***||

aaa|||, |bbb||, ||ccc|, |||ddd,

aa|b||, a|bb||, ...

...

a|b|c|, a|b||d, a||c|d, |b|c|d. *||*|*.

observe: We can fully recover the multisets just from where the 4-1=3 bars and the *'s (placeholders for letters) are.

there need to be k many.

Upshot: There is a bijection from (star-and-bars diagrams with $\overset{|X|}{n-1}$ bars and k $*$'s) to (the multisets of size k with elts from X).

So the number of the latter is

$$\binom{(n-1)+k}{n-1}, \quad \text{and also} \quad \binom{(n-1)+k}{k}.$$

For our problem: $\#\left(\text{multisets of size } k=3 \text{ w/ elts from } \underbrace{\{a,b,c,d\}}_{n=4}\right)$

$$= \binom{(4-1)+3}{4-1} = \binom{6}{3} = \frac{\cancel{6}\times 5\times 4}{\cancel{3\times 2\times 1}} = 20.$$

Examples. 3.19. A bag has 20 (identical) red marbles, 20 green marbles, and 20 blue marbles. You reach into the bag and grab 20 marbles. How many possible outcomes are there?

e.g. 7 reds, 8 greens, 5 blues $\leftrightarrow$ rrrrrrr|gggggggg|bbbbb

Soln: The outcomes correspond to multisets of size $k=20$ filled with $n=3$ elts corresponding to the three colors, so the desired number is $\binom{20+2}{2} = 231$.

3.20. How many nonnegative integer solns does $w+x+y+z=20$ have?

e.g. $w=4$, $x=2$, $y=7$, $z=7$. $\leftrightarrow$ ****|**|*******|*******

The solns correspond to separating 20 items into 4 regions, so there are

$$\binom{20+3}{3} = \binom{23}{3} \text{ solns.}$$

In the last problem, the condition that the solns have nonnegative numbers (allowing 0) is crucial for the application of the bars and stars method.

What about the problem

" How many <u>positive</u> integer solns are there for

$$w+x+y+z=20 \text{ ? "}$$

?

The answer is $\binom{16+3}{3}=\binom{19}{3}$.

<u>The trick</u>: By the "change of variable" $w \mapsto w'=w-1$, $x \mapsto x'=x-1$, $y \mapsto y'=y-1$, $z \mapsto z'=z-1$, we have a <u>bijection</u>

$$\left\{ \begin{array}{l} (w,x,y,z), \text{ positive int. soln} \\ \text{of } w+x+y+z=20 \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} (w',x',y',z'), \text{ nonnegative} \\ \text{int soln of } x'+y'+z'+w'=20-4=16 \end{array} \right\}$$

$$(w,x,y,z)=(4,2,7,7) \longmapsto (x',y',z',w')=(3,1,6,6)$$