Math 2001. Lecture 12.

02.09.2022.

Last time: (1) The binomial thm $\left((x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\right.$ and Pascal's triangle

(2) a comb. identity: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

Today:

- An application of (1) + (2)
- The inclusion-exclusion principle
- Worksheet on counting (permutations and combinations)

1. An application of (1) + (2)

Q: What's the expansion of $(x+y)^7$?

Recursion: The comb. identity in (2) gives a recursive way to compute binom. coeff: each entry not on the border equals the sum of its shoulders, which are on the previous row.

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}
$$

$$
\begin{aligned}
(x+y)^7 = {} & x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 \\
& + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7.
\end{aligned}
$$

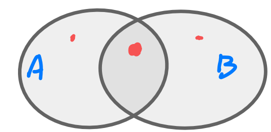
No tedious computation!

2. The Inclusion-Exclusion Principle

Prop 1: If A, B are two finite sets, then $|A \cup B| \overset{*}{=} |A| + |B| - |A \cap B|$.

Remarks:

- Reason: To get $|A \cup B|$ we should count each elt in A or B exactly once. The expression $|A| + |B|$ counts every elt in $A \setminus B$ or $B \setminus A$ once but counts every elt in $A \cap B$ twice, so $|A| + |B| - |A \cap B|$ counts the desired number.



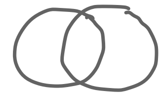
- When $A \cap B = \phi$, we have $|A \cap B| = 0$ so $*$ says $\underbrace{|A \cup B|}_{\text{disjoint}} = |A| + |B|$, recovering the addition principle. (So Prop 1 can be viewed as a generalization of the principle.)

Examples:

Recall: an integer is even $\Updownarrow$ its last digit is even.

(3.1.7) A 3-card hand is dealt off a usual 52-card deck. How many such hands are there that are all red or all face (J, Q, K)?

A: all red, B: all face



$$\underset{\text{red}}{\binom{26}{3}} + \underset{\text{face}}{\binom{3\times4}{3}} - \underset{\text{all face and all red}}{\binom{2\times3}{3}}.$$

6 red faces

(3.7.3) How many 4-digit positive numbers are there that are even or contain no zeros?

Soln: It suffices to count $A\cup B$ where $A = \{$4-digit even pos. numbers$\}$ ($\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$)

and $B = \{$4-digit pos. numbers w/ no 0's$\}$

We have $|A| = \overset{a}{9}\times \overset{b}{10}\times \overset{c}{10}\times \overset{d}{5}$. $|B| = \overset{a}{9}\times 9^3$

and $|A\cap B| = |\{$4 digit, positive, even number with no zeros$\}| = \overset{a}{9}\times \overset{d}{4}\times 9\times 9$.

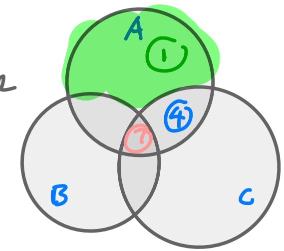
So $|A\cup B| = |A| + |B| - |A\cap B| = 9\times10\times10\times5 + 9^4 - 4\cdot 9^3$. $\square$

Prop 1 generalizes:

Prop 2: If A, B, C are three finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

We can again check the Prop by ensuring every elt in every region in the Venn diagram gets counted once eventually on the RHS.



e.g. ① : $A \setminus (B \cup C)$ $\quad +1 + 0 + 0 - 0 - 0 - 0 + 0 = 1$

④ : $A \cap C \setminus (A \cap B \cap C)$ $\quad +1 + 0 + 1 - 0 - 1 - 0 + 0 = 1$

⑦ : $A \cap B \cap C$ $\quad +1 + 1 + 1 - 1 - 1 - 1 + 1 = 1$

alternating sum.

An even more general fact: $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n| - |A_1 \cap A_2| - \cdots - |A_{n-1} \cap A_n| + |A_1 \cap A_2 \cap A_3| + \cdots$

The full generalization of Prop 1 is:

Prop 3: Let $A_1, A_2, \ldots, A_n$ be n finite sets, and let $X = \{A_1, A_2, \ldots, A_n\}$

nontrivial ~~hard!~~

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{\substack{Y \subseteq X \\ |Y| = k}} \left| \bigcap_{A \in Y} A \right| \right)$$

Challenge Ex: Prove Prop 3. (Hint: Let $a \in \bigcup_{i=1}^{n} A_i$, suppose a is contained in exactly k of the n sets $A_1, \ldots, A_n$, then show that a is counted exactly once in net effect on the RHS.)

Next time:

- Counting multisets
- more counting problems