Math 2001. Lecture 11.

02.07. 2022.

Last time:

- Counting comb.: The number of size-k subsets of a size-n set $(k \leq n)$ is

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

k factors on the num. and on the den.

- Comb. equations and their proofs.

$$\binom{n}{k} = \binom{n}{n-k}, \quad 2^n = \sum_{i=0}^{n} \binom{n}{i}$$

- Pascal's Triangle

$$1 = \binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\cdots\cdots$$

Today:

- The binomial theorem
- another comb. identity.

1. The binomial theorem

We start by computing the first few rows of Pascal's Triangle.

$$
\begin{array}{ccccccccccc}
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
\quad = \quad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & \underset{\left(\frac{4\cdot 3}{2\cdot 1}\right)}{6} & & 4 & & 1 & \\
1 & & 5 & & \underset{\frac{5\cdot 4}{2\cdot 1}}{10} & \curvearrowright & 10 & & 5 & & 1
\end{array}
$$

On the other hand, expanding $(x+y)^n$ into monomials $?\,x^k y^{n-k}$ gives ...

$$(x+y)^1 = 1\cdot x^1 y^0 + 1\cdot x^0 y^1, \quad (x+y)^2 = 1\cdot x^2 + 2\cdot xy + 1\cdot y^2,$$

$(x+y)^3 = (x+y)^2(x+y) = (x^2+2xy+y^2)(x+y) = 1\cdot x^3 + 3x^2y + 3xy^2 + 1\cdot y^3$

$(x+y)^4 = \cdots = 1\cdot x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1\cdot y^4.$

A pattern emerges ...

Thm. We have $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad \forall n \in \mathbb{Z}_{>0}.$

Pf: One way to expand $(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ copies}}$ is to sum all 2^n summands of the form $x^i y^{n-i}$, where the summand results from multiplying n factors i of which are x's selected from i of the n sets of parentheses.

It follows that the coefficient of $x^{n-k}y^k$ equals the number of ways to pick k sets of parentheses from n sets. i.e., it's $\binom{n}{k}$. $\square$

Corollary: $2^n = \sum_{k=0}^{n} \binom{n}{k} \quad \forall n \in \mathbb{Z}_{>0}.$ **Pf:** Plug $x=1, y=1$ into the theorem.

The idea used in the proof can be used in more general expansions:

Eg. · Use the binomial theorem to find the coefficient of

— x^8 in $(x+2)^{13}$

Soln: $(x+2)^{13} = (x+2)(x+2)\cdots(x+2)$, "so" x^8 has coeff. $\binom{13}{8} \cdot 2^5$.

oR: $(x+2)^{13} = \sum_{k=0}^{13} \binom{13}{k} x^k \cdot 2^{13-k}$. We are interested in the case $k=8$, for which the term is $\binom{13}{8} \cdot x^8 \cdot 2^5$, so the desired coeff. is $2^5 \cdot \binom{13}{8}$.

— $x^6 y^3$ in $(3x-2y)^9$

Soln: $(\underbrace{3x}_{X} - \underbrace{2y}_{Y})^9 = \sum_{k=0}^{9} \binom{9}{k} (3x)^k (-2y)^{9-k}$. For $k=6$, the corresponding term is $\binom{9}{6} (3x)^6 (-2y)^3 = \binom{9}{6} \cdot 3^6 \cdot (-2)^3$, so the desired coeff is $\binom{9}{6} \cdot 3^6 \cdot (-2)^3$.

2. $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

Prop: $\forall n, k \in \mathbb{Z}_{>0}$ with $n \geq k$, we have $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Eg: $\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$, $\binom{4}{2} = \binom{3}{2} + \binom{3}{1}$

Implication for Pascal's Triangle: Every non-boundary entry equals the sum of its two shoulders.

Pf of Prop:

Pf 1. (algebraic):

$$\text{RHS} = \frac{\overbrace{n(n-1)\cdots(n-k+1)}^{k \text{ terms}}}{k!} + \frac{n\cdots\overset{n-k+2}{\overset{\uparrow}{(n-(k-1)+1)}}\cdot k}{(k-1)!\cdot k}$$

↓

unenlightening!

$$= \frac{\left[\overbrace{n(n-1)\cdots(n-k+2)}^{k-1 \text{ terms}}\right]\overset{n+1}{\overset{\shortparallel}{(n-k+1+k)}}}{k!} = \frac{n(n-1)\cdots(n-k+2)(n+1)}{k!}$$

equals the prod of k terms going down from $(n+1)$

$$= \binom{n+1}{k} = \text{LHS}. \quad \checkmark$$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

pf 2 (combinatorial): Consider selecting k elts from a set $\{a_1, a_2, \cdots, a_n, a_{n+1}\}$ of $n+1$ things. There are $\binom{n+1}{k}$ ways to do this.

On the other hand, we note that to select the k elts we can either select them all from the subset $\{a_1, \ldots, a_n\}$ or select $k-1$ elts from $\{a_1, \ldots, a_n\}$ and include a_{n+1}. There are $\binom{n}{k}$ and $\binom{n}{k-1}$ for these kinds of selections, respectively. It follows that

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \quad \text{as desired.} \qquad \square$$

Next time:
- the inclusion-exclusion principle
- worksheet on counting.