Math 2001. Lecture 10.

02.02.2022.

Last time:

- Counting problems
- factorials: $0! = 1$, $1! = 1$, $2! = 2\times 1$, ..., $n! = n\times(n-1)\times\cdots\times 2\times 1$ $\forall n \in \mathbb{Z}_{\geq 0}$.
- permutations v.s. combinations/subsets

  permutations $\downarrow$ order matters.

  combinations/subsets $\downarrow$ order doesn't matter.

Today:

- more on permutations and combinations.
- Two combinatorial identities.
- The binomial theorem.

1. Permutations v.s. Combinations / Subsets

Notation: We denote the number of k-permutations of n objects by $P(n,k)$.

. ..., .. k-combinations of n objects by $C(n,k)$.
(sets of k objects)

Main result from last time: $P(n,k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$

$$C(n,k) \overset{*}{=} \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}$$

Def (binomial coefficient) The number $C(n,k) = \frac{n!}{k!\,(n-k)!}$ is called a binomial coefficient and often denoted by $\binom{n}{k}$ (read: "n choose k").

One explanation of (*) :

It suffices to show that $P(n,k) = C(n,k) \times k!$

This holds (by the mult. principle) since to form a k-permutation (LHS),

one can first pick our the k objects to be used in the perm ($C(n,k)$ ways) Step I

and then arrange them in order ($k!$ ways) Step II.

Some example computations:

$P(6,3) = 6 \times 5 \times 4$ ("key": make sure you have "k" things descending from n)

$$C(6,2) = \binom{6}{2} = \frac{P(6,2)}{2!} = \frac{6 \times 5}{2 \times 1} = 15$$

$$C(7,3) = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35 .$$

(Key: Both the denominator and the numerator have k factors.)

descend from n $k!$

Examples:

(a) (3.11) How many size-4 subsets does $\{1, 2, 3, \ldots, 9\}$ have?

Answer: $\binom{9}{4} = \frac{9 \times \cancel{8} \times 7 \times 6}{\cancel{4} \times 3 \times \cancel{2} \times 1} = 3 \times 7 \times 6 = 126.$

(b) (3.12) How many 5-elt subsets of $\{1, 2, \ldots, 9\}$ have exactly two even elts?

Soln: It suffices to specify which two even integers out of $\{2, 4, 6, 8\}$ to use and which 3 odd integers to use out of $\{1, 3, 5, 7, 9\}$, so there are $\binom{4}{2} \cdot \binom{5}{3} = \frac{4 \times 3}{2 \times 1} \cdot \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 6 \cdot 10 = 60.$

(c) (3.14) Take 5 cards out of a usual 52-card deck.

How many such hands are there with 2 clubs and 3 hearts?

$$\binom{13}{2}\binom{13}{3} = \frac{13\times12}{\cancel{2}\times1}\cdot\frac{13\times\cancel{12}\times11}{\cancel{3}\times\cancel{2}\times1} = 13^2\cdot12\cdot11$$

(d) (3.16) How many 7-digit binary strings (0010100, 1101110, etc) have an odd number of 1's?

Why is this half of $2^7 = 128$?

Soln: Let A_i be the set of 7-digit binary strings with i 1's for $i = 0, 1, 2, \ldots, 7$.

Then we need $|A_1 \cup A_3 \cup A_5 \cup A_7| = |A_1| + |A_3| + |A_5| + |A_7|$

Note that to specify an elt in A_i it's equivalent to specify which of the i positions have a 1, so $|A_i| = \binom{7}{i}$, so the desired number is

$$\binom{7}{1}+\binom{7}{3}+\binom{7}{5}+\binom{7}{7} = \frac{7}{1}+\frac{7\times6\times5}{3\times2\times1}+\cdots+\frac{7!}{7!} = 7+35+21+1 = 64.$$

Ex

2. Two combinatial identities

We present two combinatorial identities involving binom. coeff.

Recall that $\binom{n}{k} \overset{*}{=} \dfrac{n!}{k!\,(n-k)!}$ (e.g. $\binom{7}{2} = \dfrac{7!}{2!\,5!}$, $\binom{7}{5} = \dfrac{7!}{5!\,2!}$)

Prop 1. $\forall n \in \mathbb{Z}_{>0}$ and $0 \le k \le n$, $\binom{n}{k} = \binom{n}{n-k}$

pf 1 (algebraic): By $*$, LHS $= \dfrac{n!}{k!\,(n-k)!}$, RHS $= \dfrac{n!}{(n-k)!\,(n-(n-k))!} = \dfrac{n!}{(n-k)!\,k!}$.

It follows that LHS = RHS

pf 2 (combinatorial): To choose k objects from n objects it's equivalent to specify which $n-k$ objects not to choose, so $\binom{n}{k} = \binom{n}{n-k}$.

Prop 2: $2^n \overset{\triangle}{=} \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \quad \forall n \in \mathbb{Z}_{>0}.$

Pf 1 (algebraic): later (we'll use the binomial thm)

Pf 2 (combinatorial): Take a set $A = \{a_1, \dots, a_n\}$ with n elts.

Then we already know $\#(\text{subsets of } A) = 2^n$.

On the other hand,

$$\#(\text{subsets of } A) = \sum_{i=0}^{n} \#(\text{subsets of } A \text{ with } i \text{ elts})$$

$$= \sum_{i=0}^{n} \binom{n}{i} = \text{RHS}.$$

Thus, the two sides of $\triangle$ count the same object (subsets of A), so they are equal. $\square$

3. The binomial theorem

Let's compute some binomial coefficients $\binom{n}{k}$ for n, k small, arranged in the so-called Pascal's triangle:

> **Next time:**
> - related comb. identities
> - the binomial theorem.

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & \cdots & & \cdots & & &
\end{array}
\quad = \quad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & \cdots & & \cdots & & &
\end{array}
$$