In this worksheet, we use calculus to derive some properties of the "standard normal curve." The worksheet is divided into two parts: Review (Part A), and Exercises (Part B).

Part A: Review. You may want to recall the following facts about **integration in polar coordinates** from Calculus II. **NOTE:** if you feel comfortable with integration in polar coordinates, you can skip this section, and go directly to **Part B: Exercises** below.

Suppose you have a region R in the xy plane, and you want to integrate some function $f(x, y)$ over that region. That is, suppose you want to compute the definite integral

$$I = \iint_R f(x, y)\, dx\, dy. \qquad (*)$$

Sometimes, such an integral may be easier to do it you **switch to polar coordinates.** This means you replace x and y by their polar coordinate representations

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

where r is the distance from the origin to the point (x, y), and θ is the angle (between 0 and 2π) that the point makes with the positive x-axis.

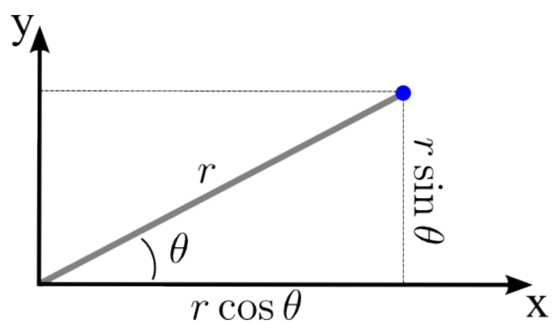

Figure 1. Polar coordinates

There are *two important things* to keep in mind when switching to polar coordinates:

1. You need to not only replace x and y by $r\cos\theta$ and $r\sin\theta$ in $f(x, y)$, but you also need to replace the "measure" $dx\, dy$ by $r\, dr\, d\theta$.

   In other words, we have

$$I = \iint_R f(x, y)\, dx\, dy = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta. \qquad (**)$$

2. You need to rewrite R in polar coordinates as well. That is: in the original formula $(*)$, R will typically be described in Cartesian (that is, xy) coordinates. To do the polar coordinate integral in $(**)$, you'll need to express R in polar coordinates.

OK, on to the Exercises.

Part B: Exercises. Note: you can do these exercises directly on this worksheet, or hand in your answers on your own (physical or virtual) paper. If you use separate paper, then for Exercise 1, which is a “fill in the blanks” exercise, you can just supply the words or formulas that go in the blanks. For Exercise 2, please show your work.

Exercise 1. As a warm-up, we’ll evaluate the integral

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} x\,(x^2+y^2)^4 \,dx\,dy,$$

by switching to polar coordinates.

Solution. (Fill in the red blanks, there are ten of them.) First, let’s work out the region of integration, in polar coordinates. We are integrating over the set

$$R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\}.$$

It’s clear what the condition $0 \le x \le 1$ means, but what about the condition $0 \le y \le \sqrt{1-x^2}$? This means we’re looking at y values between the x axis (which is the line $y = 0$) and the curve $y =$ ________________. But note that squaring both sides of $y = \sqrt{1-x^2}$ gives $y^2 = \left(\sqrt{1-x^2}\right)^2 = 1 - x^2$, so $x^2 + y^2 =$ ____________. The latter is the formula for the circle of radius one, centered at the point ______________. So we’re integrating over the region that’s under that circle, over the y axis, and satisfying $0 \le x \le 1$. That is, we’re integrating over this quarter-circle:

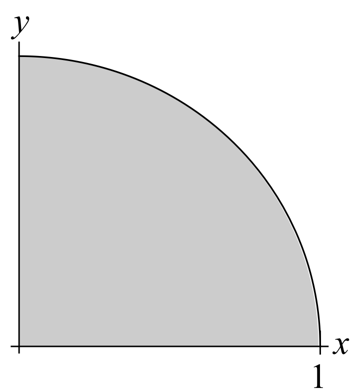

Figure 2. The region $R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\}$

In polar coordinates, this region may be described as $R = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}\}$. Putting this information, together with the substitutions $x = r\cos\theta$, $y =$ ________________, and $dx\,dy = r\,dr\,d\theta$, into our integral in x and y, we find that

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} x\left(x^2+y^2\right)^4 dx\,dy = \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} r\cos\theta\left((r\cos\theta)^2 + (r\sin\theta)^2\right)^4 r\,dr\,d\text{\_\_\_\_\_\_\_\_}.$$

Now, since $\cos^2\theta + \sin^2\theta = 1$ always, we find that

$$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2(\cos^2\theta + \sin^2\theta) = \underline{\qquad\qquad\qquad},$$

so

$$\int_{r=0}^{1}\int_{\theta=0}^{\pi/2} r\cos\theta\left((r\cos\theta)^2 + (r\sin\theta)^2\right)^4 r\,dr\,d\theta = \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} r\cos\theta\left(r^2\right)^4 r\,dr\,d\theta$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} \cos\theta\, r^{10}\,dr\,d\theta.$$

Now note that the integral on the right breaks up into two separate integrals: one in the variable r, and the other in the variable θ. The first of these integrals equals

$$\int_{r=0}^{1} r^{10}\,dr = \left.\frac{r^{11}}{11}\right|_0^1 = \frac{1}{11}(1^{11} - 0^{11}) = \frac{1}{11}.$$

The second integral equals

$$\int_{\theta=0}^{\pi/2} \cos\theta\,d\theta = \sin\theta\Big|_0^{\pi/2} = \sin(\pi/2) - \sin(\underline{\qquad\qquad\qquad}) = \underline{\qquad\qquad\qquad}.$$

So the final result is

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} x\left(x^2 + y^2\right)^4 dx\,dy = \frac{1}{11} \times \underline{\qquad\qquad\qquad} = \underline{\qquad\qquad\qquad}.$$

Exercise 2. For this Exercise, answer parts (a)(b)(c)(d) below, and please show all your work. You can write your answers in the blank spaces provided on this worksheet, or on separate (physical or virtual) paper.

The purpose of this Exercise is to show that the standard normal curve

$$f_{0,1}(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2},$$

with domain equal to the entire real line, really IS a pdf. What does this mean? Well, recall that, for a pdf, probability equals area, so the area under a pdf must equal 1. So we want to show that this is true for $f_{0,1}(x)$. That is, we want to show that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 1,$$

or in other words, that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}. \tag{$\bigstar$}$$

To do this, we're going to use a trick: we're going to multiply the integral by itself, to get a double integral, and then switch to polar coordinates.

(a) Let's call the integral that we're trying to evaluate, in (), I:

$$I = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx.$$

Explain why

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy.$$ ()

Hint: break the integral on the right up into a product of two integrals, using the fact that $e^{a+b} = e^a\, e^b$.

(b) Write the integral I^2 defined by () as an integral in polar coordinates. Hint: I^2 is an integral over the entire xy plane. In polar coordinates, this plane can be described as $\{(r, \theta) : 0 \leq r < \infty,\ 0 \leq \theta \leq 2\pi\}$. **Don't forget that $dx\,dy$ becomes $r\,dr\,d\theta$!!** Also, you should simplify the quantity $(r\cos\theta)^2 + (r\sin\theta)^2$ that you get in your exponent, using the fact that $\cos^2\theta + \sin^2\theta = 1$.

You don't need to evaluate the (r, θ) integral that you get—yet.

(c) If you did part **(b)** correctly, the integral that you get, in r and θ, should break up into an integral in r times an integral in θ. Evaluate each of these integrals using calculus (not using Wolfram Alpha, or a calculator, etc.). Please show all work. Some hints:

- **(i)** The integral in θ should be straightforward. Hint: you should get 2π for this integral.
- **(ii)** To do the integral in r, make the u-substitution $u = -r^2/2$. Hint: you should get 1 for this integral.

(d) To summarize your work on this problem, answer these questions: what is I^2 (as a number)? Use this information to answer: what is I? (Hint: I has to be positive, since it's the integral of a function that's always positive.) What is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \ ?$$

Finally, what is the area under the graph of $f_{0,1}$ (over its entire domain)? Note: some of the questions in this part may have the same answer.