PART I. Basic probability.

1. Permutations and combinations.

(a) (Permutations.) The number of ways of picking r objects out of n objects, keeping track of order, is

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!}.$$

This number is sometimes called "n pick r," written ${}_nP_r$.

(b) (Combinations.) The number of ways of choosing r objects out of n objects, without keeping track of order, is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)}{r!} = \frac{n!}{r!(n-r)!}.$$

This number is sometimes called "n choose r," written ${}_nC_r$ or $\binom{n}{r}$.

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2. Probability axioms.

(a) $P(A) \geq 0$ for any event A.

(b) $P(S) = 1$, where S is the sample space.

(c) If the events $A_1, A_2, A_3, A_4, \ldots$ are mutually exclusive (no two of them can happen together), then

$$P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + P(A_4) + \cdots.$$

(The list $A_1, A_2, A_3, A_4, \ldots$ could be finite or infinite.)

(d) If all outcomes in a sample space S are equally likely, and $n(A)$ denotes the number of outcomes in the event A, then

$$P(A) = \frac{n(A)}{n(S)}$$

(assuming the sample space is a finite set).

(e) If all outcomes in a sample space S are equally likely, and $n(A)$ denotes the number of outcomes in the event A, then

$$P(A) = \frac{n(A)}{n(S)}$$

(assuming the sample space is a finite set).

(f) If events $A_1, A_2, A_3, A_4, \ldots$ are independent (they don't affect each other), then

$$P(A_1 A_2 A_3 A_4 \cdots) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4) \cdots.$$

(The list $A_1, A_2, A_3, A_4, \ldots$ could be finite or infinite.)

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3. Probability rules.

(a) For any event A,

$$P(A) = 1 - P(A^c),$$

where A^c denotes the complement of A (meaning all outcomes in the sample space except those in A).

(b) For any events A and B (not necessarily mutually exclusive), we have

$$P(A \cup B) = P(A) + P(B) - P(AB).$$

(c) For any events A, B, and C (not necessarily mutually exclusive), we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC).$$

PART II. Conditional probability.

1. Formulas for $P(A|B)$.

(a) Suppose all events in your sample space are equally likely. Then for any events A and B, we have

$$P(A|B) = \frac{n(AB)}{n(B)}.$$

(b) Given any events A and B, we have

$$P(A|B) = \frac{P(AB)}{P(B)}$$

(whether or not events are equally likely).

2. Formulas for $P(AB)$.

(a) Given any events A and B, we have

$$P(AB) = P(B) \cdot P(A|B).$$

(b) (Generalization.) Given any events A, B, and C, we have

$$P(ABC) = P(C) \cdot P(B|C) \cdot P(A|BC).$$

(c) (Further generalization.) Given any finite or infinite list of events, we have

$$P(\text{all events happen}) = P(\text{first one happens}) \cdot P(\text{second happens given that first does}) \cdot P(\text{third does given that first two do}) \cdot P(\text{fourth does given that first three do}) \cdots .$$

3. Conditional probability and mutually exclusive events.

(a) For any events A and B,

$$P(A) = P(B)P(A|B) + P(B^c)P(A|B^c)$$

(again, B^c denotes the complement of B).

(b) (Generalization.) Suppose $B_1, B_2, B_3, \ldots, B_n$ are mutually exclusive events, and the event A can only happen if one of the events B_i happens. Then

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3) + \cdots + P(B_n)P(A|B_n).$$
