HW 8: Solutions

Section 1 Exercises.

Exercise 1.1. You pay \$10 to play the following game. You toss a fair, six-sided die. If the die lands on a 1 or a 4, you choose two marbles at random from a jar containing 4 red marbles and 2 blue marbles. If the die lands 2, 3, or 5, you choose two marbles at random from a jar containing 2 red marbles and 4 blue marbles. If the die lands 6, you choose two marbles at random from a jar containing 5 red marbles and 1 blue marble. You are then awarded \$20 if you end up with two red marbles; \$10 if you end up with one, and \$0 if you end up with no red marbles.

(a) Find the probability mass function for your payoff X (meaning how much you receive minus the \$10 put in to play).

(b) What is your expected payoff (meaning how much you receive minus the \$10 put in to play) from this game?

(c) What are the variance and standard deviation of your payoff?

SOLUTION: First of all note that, originally, this problem said "you choose a marble at random" instead of "you choose two marbles at random." That was a typo. I meant two marbles. I made the correction before most of you read the problem, so most of you, presumably, answered the correct question. But if you did the problem assuming that you were only choosing one marble, you did not lose credit. The solution below assumes that two marbles are chosen.

(a) X can equal -10 dollars (if you get no red marbles), \$0 (if you get one red marble), or \$10 (if you get two red marbles). The probabilities of these events are as follows:

$$\begin{aligned}P(X=-10) &= P(\text{no red marbles}) = P(\text{two blue marbles})\\ &= P(\text{die lands 1 or 4})\cdot P(\text{two blue given die lands 1 or 4})\\ &+ P(\text{die lands 2, 3, or 5})\cdot P(\text{two blue given die lands 2, 3, or 5})\\ &+ P(\text{die lands 6})\cdot P(\text{two blue given die lands 6})\\ &= \tfrac{1}{3}\big(\tfrac{2}{6}\cdot\tfrac{1}{5}\big)+\tfrac{1}{2}\big(\tfrac{4}{6}\cdot\tfrac{3}{5}\big)+\tfrac{1}{6}\big(\tfrac{1}{6}\cdot\tfrac{0}{5}\big)=\tfrac{2}{9}=0.222\ldots.\end{aligned}$$

Now to compute $P(X=0)$, we need to consider that either the first marble is red and the second is blue, the other way around. So

$$\begin{aligned}P(X=0) &= P(\text{one red marble})\\ &= P(\text{die lands 1 or 4})\cdot P(\text{one red given die lands 1 or 4})\\ &+ P(\text{die lands 2, 3, or 5})\cdot P(\text{one red given die lands 2, 3, or 5})\\ &+ P(\text{die lands 6})\cdot P(\text{one red given die lands 6})\\ &= \tfrac{1}{3}\big(\tfrac{4}{6}\cdot\tfrac{2}{5}+\tfrac{2}{6}\cdot\tfrac{4}{5}\big)+\tfrac{1}{2}\big(\tfrac{2}{6}\cdot\tfrac{4}{5}+\tfrac{4}{6}\cdot\tfrac{2}{5}\big)+\tfrac{1}{6}\big(\tfrac{5}{6}\cdot\tfrac{1}{5}+\tfrac{1}{6}\cdot\tfrac{5}{5}\big)=\tfrac{1}{2}=0.5.\end{aligned}$$

Finally,

$$\begin{aligned}P(X=10) &= P(\text{two red marbles})\\ &= \tfrac{1}{3}\big(\tfrac{4}{6}\cdot\tfrac{3}{5}\big)+\tfrac{1}{2}\big(\tfrac{2}{6}\cdot\tfrac{1}{5}\big)+\tfrac{1}{6}\big(\tfrac{5}{6}\cdot\tfrac{4}{5}\big)=\tfrac{5}{18}=0.2777\ldots.\end{aligned}$$

(b)

$E(X) = -10 \cdot P(X=-10) + 0 \cdot P(X=0) + 10 \cdot P(X=10) = -10 \cdot \frac{2}{9} + 0 \cdot \frac{1}{2} + 10 \cdot \frac{5}{18} = \frac{5}{9} = 0.555\ldots.$

(c)

$\text{var}(X) = (-10 - \frac{5}{9})^2 \cdot P(X=-10) + (0 - \frac{5}{9})^2 \cdot P(X=0) + (10 - \frac{5}{9})^2 \cdot P(X=10) = 5.10974,$

so $\text{std}(X) = \sqrt{5.10974} = 2.26047$.

Exercise 1.2. There are twelve people in a room, and each person chooses a digit from 0 through 9 at random. Then each person writes down their digit, on the same piece of paper, one after the other, so that the result is a string of twelve digits. Let X be the number of times the same digit appears twice in a row. (For example, if the string is 133304994437, then $X = 4$; the string "333" counts as two pairs of repeated digits.)

(a) Find the probability mass function for X.

(b) Find $E(X)$.

(c) What are $\text{var}(X)$ and $\text{std}(X)$?

Hint: Let X_1 equal 1 if the first two digits are the same, and 0 if not; let X_2 equal 1 if the second and third digits are the same, and 0 if not; ...let X_{11} equal 1 if the eleventh and twelfth digits are the same, and 0 if not. Then $X = X_1 + X_2 + \cdots + X_{11}$. Now use Theorem 1.4.

SOLUTION: (a) (Oops, this exercise really should have been in the next section, since it uses binomial probabilities.) Suppose there are k times that the same digit appears twice in a row. There are $\binom{11}{k}$ ways to place these k pairs of consecutive digits, and for each such way, the probability of actually having k pairs match and the rest not match is $0.1^k \cdot (0.9)^{11-k}$. So, for $0 \le k \le 11$,

$$P(X = k) = \binom{11}{k} \cdot 0.1^k \cdot 0.9^{11-k}.$$

(b) Let X_1 through X_{11} be as in the hint. Note that $E(X_1) = \frac{1}{10}$: the first digit could be anything, and then the probability that the next digit matches the first is $\frac{1}{10}$. Similarly, $E(X_j) = \frac{1}{10}$ for $j = 1, 2, 3, \ldots$. So

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_{11}) = 11 \cdot \tfrac{1}{10} = 1.1.$$

(c) Note that the first two digits matching is independent of the second and third matching, which is independent of the third and fourth matching, and so on. Also note that, since X_1 equals 1 with probability $\frac{1}{10}$ and equals 0 with probability $\frac{9}{10}$, we have

$$\text{var}(X_1) = (1 - \tfrac{1}{10})^2 \cdot P(X_1 = 1) + (0 - \tfrac{1}{10})^2 \cdot P(X_1 = 0) = \tfrac{9}{100} = 0.09,$$

The variance is the same for all of the other X_j's, so

$$\text{var}(X) = \text{var}(X_1) + \text{var}(X_2) + \cdots + \text{var}(X_{11}) = 11 \cdot 0.09 = 0.99.$$

So $\text{std}(X) = \sqrt{0.99} = 0.995$.

Section 2 Exercises.

Exercise 2.1. A 40% three-point field goal shooter attempts five three-point shots in a game.

(a) Find the probability mass function for the number X of three-point shots made out of the five. Confirm that your probabilities add up to one.

(b) Find the probability that the player makes at least two three-point shots, out of the five taken. Hint: it might be easier to first compute the probability that they make fewer than two.

(c) Find the expected number of three-point shots made. Please compute this in two ways: (i) using the method of Example 2.4 above (that is, using the definition of expected value, and the probabilities that you computed in part (a) of this exercise), and (ii) using Theorem 2.5. Of course, you should get the same answer either way.

(d) Use Theorem 2.5 to find $\text{var}(X)$ and $\text{std}(X)$.

SOLUTION: (a)

$$\begin{aligned} P(X=0) &= \tbinom{5}{0} \cdot 0.4^0 \cdot 0.6^5 = 0.0778, \\ P(X=1) &= \tbinom{5}{1} \cdot 0.4^1 \cdot 0.6^4 = 0.2592, \\ P(X=2) &= \tbinom{5}{2} \cdot 0.4^2 \cdot 0.6^3 = 0.3456, \\ P(X=3) &= \tbinom{5}{3} \cdot 0.4^3 \cdot 0.6^2 = 0.2304, \\ P(X=4) &= \tbinom{5}{4} \cdot 0.4^4 \cdot 0.6^1 = 0.0768, \\ P(X=5) &= \tbinom{5}{5} \cdot 0.4^5 \cdot 0.6^0 = 0.0102. \end{aligned}$$

Note that these numbers do add up to one.

(b)

$$P(X \geq 2) = 1 - P(X < 2) = 1 - P(X=0) - P(X=1) = 1 - 0.0778 - 0.2592 = 0.6630 = 66.3\%.$$

(c) First way:

$$\begin{aligned} E(X) &= \sum_{k=0}^{5} k \cdot P(X=k) \\ &= 0 \cdot 0.0778 + 1 \cdot 0.2592 + 2 \cdot 0.3456 + 3 \cdot 0.2304 + 4 \cdot 0.0768 + 5 \cdot 0.0102 = 1.998. \end{aligned}$$

Second way: $E(X) = n\,p = 5 \cdot 0.4 = 2$. (There's some roundoff error in the first way, but apart from that they're equal.)

(d) By Theorem 2.5, $\text{var}(X) = n\,p\,(1-p) = 5 \cdot 0.4 \cdot 0.6 = 1.2$, so $\text{std}(X) = \sqrt{1.2} = 1.095$.

Exercise 2.2. Consider the following game. You pay \$10, and pick a number from 1 through 6. A fair die is rolled three times. You are awarded \$0 if your number does not come up at all in the three rolls; you are awarded \$20 if your number comes up once, \$30 if it comes up twice, and \$40 if it comes up all three times.

Let X be the number of times your chosen number comes up. Then X is the number of successes in three trials of a binomial experiment, with $P(\text{success}) = 1/6$.

(a) Find the probability mass function for X.

(b) Should you play the game? Hint: your expected payoff, in dollars, is

$$-10 \cdot P(X=0) + 10 \cdot P(X=1) + 20 \cdot P(X=2) + 30 \cdot P(X=3).$$

SOLUTION: (a)

$$P(X=0) = \binom{3}{0} \cdot \left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^3 = 0.5787,$$
$$P(X=1) = \binom{3}{1} \cdot \left(\frac{1}{6}\right)^1 \cdot \left(\frac{5}{6}\right)^2 = 0.3472,$$
$$P(X=2) = \binom{3}{2} \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^1 = 0.0694,$$
$$P(X=3) = \binom{3}{3} \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^0 = 0.0047.$$

(b)

$$-10 \cdot P(X=0) + 10 \cdot P(X=1) + 20 \cdot P(X=2) + 30 \cdot P(X=3) = -0.786.$$

The expected payoff is negative, so you should not play the game.

Section 3 Exercises.

Exercise 3.1. A basketball player hits 60% of their shots. Suppose all of their shots are independent of each other. Suppose this player takes 25 shots in a game. Let X be the number of shots in this game, out of 25, that the player makes.

(a) Using Theorem 2.3 (that is, *do not* use the normal approximation), compute $P(X=14)$, $P(X=15)$, and $P(X=16)$ directly.

(b) Use your results from part (a) of this exercise to compute $P(14 \leq X \leq 16)$.

(c) In the current setting, are the hypotheses of Theorem 3.1 above satisfied?

(d) Use Theorem 3.1 to approximate $P(14 \leq X \leq 16)$. Use the method of Example 3.2 above. (That is, convert your normal probability to standard normal.) **You may use the fact that, if Z is standard normal, then**

$$P(-0.612 < Z < 0.612) = 0.4597.$$

How do your results compare to those of part (b) of this exercise, above?

SOLUTION: (a) We compute that

$$P(X=14) = \binom{25}{14} \cdot 0.6^{14} \cdot (0.4)^{25-14} = 0.1465,$$

and similarly, $P(X=15) = 0.1612$, $P(X=16) = 0.1511$.

(b)

$$P(14 \leq X \leq 16) = 0.1465 + 0.1612 + 0.1511 = 0.4588.$$

(c) Yes, because $n\,p = 25 \cdot 0.6 = 15 \geq 10$, and $n\,(1-p) = 25 \cdot 0.4 = 10 \geq 10$.

(d) In the present case, the mean is $n\,p = 15$ and the standard deviation is $\sqrt{n\,p\,(1-p)} \approx 2.45$. So

$$\begin{aligned} P(14 \le X \le 16) &\approx P(13.5 \le Y \le 16.5) \\ &= P\left(\frac{13.5-15}{2.5} < \frac{Y-15}{2.5} < \frac{16.5-15}{2.5}\right) \\ &= P\left(-0.612 < \frac{Y-15}{2.5} < 0.612\right) = 0.4597. \end{aligned}$$

This agrees with the answer to part (b) to two decimal places.

Exercise 3.2. A fair die is rolled 180 times. Let X be the number of times a 6 comes up.

(a) Use Theorem 3.1 above to approximate $P(X \le 35)$. **You may use the fact that, if Z is standard normal, then**
$$P(Z < 1.1) = 0.8643.$$

(b) Use your answer to part (a) of this exercise to approximate $P(X > 35)$.

(c) Now consider the following game: with the above 180 rolls of a fair die, you win \$5 if a 6 comes up at most 35 times, and lose \$30 if a 6 comes up more than 35 times. What are your expected winnings?

SOLUTION: (a) The mean here is $n\,p = 180\cdot\frac{1}{6} = 30$ and the standard deviation is $\sqrt{n\,p\,(1-p)} = 5$. So

$$\begin{aligned} P(X \le 35) &\approx P(Y < 35.5) \\ &= P\left(\frac{Y-30}{5} < \frac{35.5-30}{5}\right) = P\left(\frac{Y-30}{5} < 1.1\right) = 0.8643. \end{aligned}$$

(b) $P(X > 35) = 1 - P(X \le 35) = 1 - 0.8643 = 0.1357$.

(c) The expected winnings are, approximately,

$$-30 \cdot P(X > 35) + 5 \cdot P(X \le 35) = -30 \cdot 0.1357 + 5 \cdot 0.8643 = 0.2505.$$

You should play; on average, you'd expect to win about 25 cents.