Some notes on random variables: expected value, variance, standard deviation, the binomial distribution, and the normal approximation to the binomial distribution

1 Expected value, variance, and standard deviation

We begin with some definitions.

Definition 1.1. If X is a *discrete* random variable (one that takes on only finitely many or countably many values), then we define:

(a) The *expected value* $E(X)$ by

$$E(X) = \sum_x x \cdot P(X = x),$$

where the sum is taken over all possible values x of X.

We often write $\mu(X)$, or simply μ, for $E(X)$.

(b) The *variance* $\mathrm{var}(X)$ by

$$\mathrm{var}(X) = E\big((X-\mu)^2\big) = \sum_x (x-\mu)^2 \cdot P(X = x),$$

where μ denotes the expected value of X, as defined directly above. (And, as before, the sum is over all possible values of X.)

We often write $\sigma^2(X)$, or simply σ^2, for $\mathrm{var}(X)$.

(c) The *standard deviation* $\mathrm{std}(X)$ by

$$\mathrm{std}(X) = \sqrt{\mathrm{var}(X)},$$

where $\mathrm{var}(X)$ denotes the variance of X, as defined directly above.

We often write $\sigma(X)$, or simply σ, for $\mathrm{std}(X)$.

Remark: the above definitions of μ and σ are essentially the same as the definitions, given earlier, of mean and standard deviation of a data set X. See, for example, Exercise 4 from Homework #5. (There's a slight discrepancy between the above definition of σ and the one given earlier, and this has to do with the fact that the earlier definition has an $n-1$, instead of an n, in the denominator. But let's not worry about that: if n is large, then the difference between $1/n$ and $1/(n-1)$ is small.)

Also: note that $\mu = E(X)$ and $\sigma = \mathrm{std}(X)$ have the same units as X (though σ^2 does not).

Finally: one can also define expected value, variance, and standard deviation of continuous random variables, but we'll leave that for another day (though we've already done some of this in the context of *normal* random variables).

Example 1.2. You pay \$4 to play the following game. You toss an unfair coin, with P(heads)= 1/4. If the coin lands heads, you choose two marbles at random from a jar containing 4 red marbles and 2 blue marbles. If the coin lands tails, you choose two marbles at random from a jar containing 3 red marbles and 3 blue marbles. You are then awarded \$20 if you end up with two red marbles; otherwise, you receive \$0.

(a) Find the probability mass function for your payoff X (meaning how much you receive minus the \$4 put in to play).

SOLUTION: The payoff X is either -4 dollars or 16 dollars. We have

$$\begin{aligned} P(X = 16) &= P(\text{two red marbles}) \\ &= P(\text{coin lands heads}) \cdot P(\text{two red marbles given coin lands heads}) \\ &+ P(\text{coin lands tails}) \cdot P(\text{two red marbles given coin lands tails}) \\ &= \frac{1}{4} \cdot \frac{4}{6} \cdot \frac{3}{5} + \frac{3}{4} \cdot \frac{3}{6} \cdot \frac{2}{5} = \frac{1}{4} = 0.25. \end{aligned}$$

Since $X = -4$ and $X = 16$ are the only two possibilities, we must have

$$P(X = -4) = 1 - P(X = 16) = 1 - 0.25 = 0.75.$$

(b) What is your expected payoff (meaning how much you receive minus the \$4 put in to play) from this game?

SOLUTION:

$$\mu = E(X) = -4 \cdot P(X = -4) + 16 \cdot P(X = 16) = -4 \cdot \frac{3}{4} + 16 \cdot \frac{1}{4} = \frac{4}{4} = 1$$

dollar.

(c) What are the variance and standard deviation of your payoff?

SOLUTION:

$$\sigma^2 = \text{var}(X) = (-4 - \mu)^2 \cdot P(X = -4) + (16 - \mu)^2 \cdot P(X = 16) = (-5)^2 \cdot 0.75 + 15^2 \cdot 0.25 = 75.$$

So $\sigma = \sqrt{75} \approx 8.66$ dollars.

Example 1.3. A fair die is tossed two times. Let the random variable X be the largest of the two outcomes. What is the probability mass function of X? What are $E(X)$, $\text{var}(X)$, and $\text{std}(X)$?

SOLUTION: We have already computed the probability mass function for X, in Homework Assignment #4, Problem 9.1. For example, we computed there that

$$P(X = 4) = \frac{n(\{14, 41, 24, 42, 34, 43, 44\})}{36} = \frac{7}{36}.$$

(Here, for example, the outcome "24" means the first die showed a 2 and the second showed a 4.) Instead of repeating all of those calculations, we summarize:

$$P(X=1)=\frac{1}{36}, \qquad P(X=2)=\frac{3}{36}, \qquad P(X=3)=\frac{5}{36},$$
$$P(X=4)=\frac{7}{36}, \qquad P(X=5)=\frac{9}{36}, \qquad P(X=6)=\frac{11}{36}.$$

So

$$\mu=E(X)=1\cdot\frac{1}{36}+2\cdot\frac{3}{36}+3\cdot\frac{5}{36}+4\cdot\frac{7}{36}+5\cdot\frac{9}{36}+6\cdot\frac{11}{36}=\frac{161}{36}\approx 4.47.$$

Similarly

$$\mathrm{var}(X)=(1-\mu)^2\cdot\frac{1}{36}+(2-\mu)^2\cdot\frac{3}{36}+\cdots+(6-\mu)^2\cdot\frac{11}{36}\approx 1.97,$$

so $\mathrm{std}(X)=\sqrt{\mathrm{var}(X)}\approx 1.40$.

The expected value, variance, and standard deviation have some nice properties, summarized in the following theorem (which we won't prove).

Theorem 1.4. (a) For any random variables X and Y, we have

$$E(X+Y)=E(X)+E(Y).$$

(b) If X and Y are **independent** random variables, then

$$\mathrm{var}(X+Y)=\mathrm{var}(X)+\mathrm{var}(Y).$$

Remark: It is *not* true, in general, that $\mathrm{std}(X+Y)=\mathrm{std}(X)+\mathrm{std}(Y)$, even if X and Y are independent. Instead one has, by part (b) of the above theorem and by the fact that $\mathrm{std}(Z)=\sqrt{\mathrm{var}(Z)}$ for any random variable Z, that

$$\mathrm{std}(X+Y)=\sqrt{\mathrm{var}(X+Y)}=\sqrt{\mathrm{var}(X)+\mathrm{var}(Y)}=\sqrt{(\mathrm{std}(X))^2+(\mathrm{std}(Y))^2}.$$

Or, writing $\sigma(Z)$ for $\mathrm{std}(Z)$,

$$\sigma(X+Y)=\sqrt{\sigma^2(X)+\sigma^2(Y)}.$$

Again, this assumes that X and Y are independent. if not, Theorem 1.4 need not apply.

Example 1.5. Suppose 10 fair dice are rolled, and the way each die lands is independent of the other dice. Find the mean, variance, and standard deviation of the total of the numbers showing on the 10 dice.

SOLUTION: Let X_j, for $1\le j\le 10$, denote the number showing on the jth die. Since the die is fair, each number has probability $1/6$ of coming up, so the expected value of the number showing up on the jth die is

$$\mu_j=E(X_j)=1\cdot\frac{1}{6}+2\cdot 16+\cdots+6\cdot 1/6=\frac{21}{6}=3.5.$$

Then

$$\mathrm{var}(X_j)=(1-\mu_j)^2\cdot\frac{1}{6}+(2-\mu_j)^2\cdot\frac{1}{6}+\cdots+(6-\mu_j)^2\cdot\frac{1}{6}=\frac{35}{12}\approx 2.9167,$$

and $\mathrm{std}(X_j) = \sqrt{\mathrm{var}(X_j)} \approx 1.708$.

Now let X denote the sum of the 10 numbers coming up. Then $X = X_1 + X_2 + \cdots + X_{10}$, so by Theorem 1.4 (extended to 10 random variables),

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_{10}) = 10 \cdot 3.5 = 35;$$

$$\mathrm{var}(X) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + \cdots + \mathrm{var}(X_{10}) = 10 \cdot 2.916 = 29.167;$$

and $\mathrm{std}(X) = \sqrt{29.167} \approx 5.401$.

Section 1 Exercises.

Exercise 1.1. You pay \$10 to play the following game. You toss a fair, six-sided die. If the die lands on a 1 or a 4, you choose two marbles at random from a jar containing 4 red marbles and 2 blue marbles. If the die lands 2, 3, or 5, you choose two marbles at random from a jar containing 2 red marbles and 4 blue marbles. If the die lands 6, you choose two marbles at random from a jar containing 5 red marbles and 1 blue marble. You are then awarded \$20 if you end up with two red marbles; \$10 if you end up with one, and \$0 if you end up with no red marbles.

(a) Find the probability mass function for your payoff X (meaning how much you receive minus the \$10 put in to play).

(b) What is your expected payoff (meaning how much you receive minus the \$10 put in to play) from this game?

(c) What are the variance and standard deviation of your payoff?

Exercise 1.2. There are twelve people in a room, and each person chooses a digit from 0 through 9 at random. Then each person writes down their digit, on the same piece of paper, one after the other, so that the result is a string of twelve digits. Let X be the number of times the same digit appears twice in a row. (For example, if the string is 133304994437, then $X = 4$; the string "333" counts as two pairs of repeated digits.)

(a) Find the probability mass function for X.

(b) Find $E(X)$.

(c) What are $\mathrm{var}(X)$ and $\mathrm{std}(X)$?

Hint: Let X_1 equal 1 if the first two digits are the same, and 0 if not; let X_2 equal 1 if the second and third digits are the same, and 0 if not; . . . let X_{11} equal 1 if the eleventh and twelfth digits are the same, and 0 if not. Then $X = X_1 + X_2 + \cdots + X_{11}$. Now use Theorem 1.4.

2 The binomial distribution

A *binomial* experiment is one with only two possible outcomes. We generally call one of these outcomes a *success* and the other a *failure*. We typically denote the probability of success by p.

Unless otherwise specified, we will always assume that all trials of a given binomial experiment are independent of each other.

The following theorem is straightforward.

Theorem 2.1. Let X denote the number of successes in a single trial of a binomial experiment with $P(\text{success}) = p$. Then

$$E(X) = p, \qquad \text{var}(X) = p(1-p), \qquad \text{std}(X) = \sqrt{p(1-p)}.$$

Proof. If X is as described, then X takes the value 1 with probability p, and takes the value 0 with probability $1-p$, and takes no other values, so

$$\mu = E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) = 0 \cdot (1-p) + 1 \cdot p = p.$$

But then

$$\begin{aligned}
\text{var}(X) &= (0-\mu)^2 \cdot P(X=0) + (1-\mu^2) \cdot P(X=1) \\
&= (0-p)^2 \cdot (1-p) + (1-p)^2 \cdot p \\
&= p^2 \cdot (1-p) + (1-p)^2 \cdot p \\
&= p^2 - p^3 + p - 2p^2 + p^3 = p - p^2 = p(1-p).
\end{aligned}$$

Finally, $\text{std}(X) = \sqrt{\text{var}(X)} = \sqrt{p(1-p)}$.

$\square$

Example 2.2. If X denotes the number of shots made by an 80% free-throw shooter in a single attempt, find $E(X)$, $\text{var}(X)$, and $\text{std}\, X$.

SOLUTION: by Theorem 2.1, we have $E(X) = p = 0.8$,

$$\text{var}(X) = 0.8 \cdot (1-0.8) = 0.8 \cdot 0.2 = 0.16,$$

and $\text{std}(X) = \sqrt{0.16} = 0.4$.

Often we will want to understand what things look like if we repeat a binomial experiment n times. One question we might want to ask in such a context is: what is the probability that exactly k of these n trials of our binomial experiment will result in success? Here $0 \leq k \leq n$.

Here is how we might answer this question: first note that, since $P(\text{success}) = p$, we also have $P(\text{failure}) = 1-p$. Next: there are $\binom{n}{k}$ different ways to slot the k successes in among the n trials. For each such way of choosing where to put our k successes, what is the probability that there actually is a success in each of these places, and a failure in each other place? Well, the probability of success in any given slot is p, so the probability of having a success in k different slots is p^k. Similarly, the probability of having a failure in the remaining $n-k$ slots is $(1-p)^{n-k}$. Putting all of this together, we have the following.

Theorem 2.3. If a binomial experiment, with $P(\text{success}) = p$, is repeated n times (and, as usual, all trials are independent), then the probability of exactly k successes among those n trials is given by

$$P(k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k (1-p)^{n-k} \qquad (0 \leq k \leq n).$$

Remark: when we write P(k successes in n trials), we will always mean the probability of *exactly* k successes (rather than *at least* k, or *at most* k, or anything of that nature) among the n trials.

Example 2.4. An 80% free-throw player attempts four free throws.

(a) Find the probability mass function for the number X of free throws made out of the four.

(b) Find the probability that the player makes at least two free throws.

(c) Find the expected number of free throws made. Also find $\text{var}(X)$ and $\text{std}(X)$.

SOLUTION: (a) By Theorem 2.3, we have:

$$\begin{aligned}
P(X=0) &= \binom{4}{0}(0.8)^0(0.2)^{4-0} = 0.0016,\\
P(X=1) &= \binom{4}{1}(0.8)^1(0.2)^{4-1} = 0.0256,\\
P(X=2) &= \binom{4}{2}(0.8)^2(0.2)^{4-2} = 0.1536,\\
P(X=3) &= \binom{4}{3}(0.8)^3(0.2)^{4-3} = 0.4096,\\
P(X=4) &= \binom{4}{4}(0.8)^4(0.2)^{4-4} = 0.4096.
\end{aligned}$$

Note that the probabilities add up to 1.

(b) We compute that

$$P(X \text{ is at least two}) = P(X=2) + P(X=3) + P(X=4) = 0.1536 + 0.4096 + 0.4096 = 0.9728.$$

(c) We have

$$\begin{aligned}
E(X) &= 0 \cdot P(X=0) + 1 \cdot P(X=1) \cdots + 4 \cdot P(X=4)\\
&= 0 \cdot 0.0016 + 1 \cdot 0.0256 + 2 \cdot 0.1536 + 3 \cdot 0.4096 + 4 \cdot 0.4096 = 3.2.
\end{aligned}$$

Also,

$$\begin{aligned}
\text{var}(X) &= (0-3.2)^2 \cdot P(X=0) + (1-3.2)^2 \cdot P(X=1) \cdots + (4-3.2)^2 \cdot P(X=4)\\
&= 0.64,
\end{aligned}$$

and $\text{std}(X) = \sqrt{0.64} = 0.8$.

It's not surprising that, in the above example, $E(X) = 3.2$: if the player hits 80% of their shots then, out of 4 shots, we would expect them to hit 80% of those four, and $0.8 \times 4 = 3.2$.

In fact, we could have computed $E(X)$ more easily, and similarly, we could easily have computed $\text{var}(X)$ and $\text{std}(X)$, using the following.

Theorem 2.5. Given a binomial experiment with $P(\text{success}) = p$, let X denote the number of successes in n trials of the experiment. Then

$$E(X) = np, \qquad \text{var}(X) = np(1-p), \qquad \text{std}(X) = \sqrt{np(1-p)}.$$

Proof. For such an experiment, let X_j denote the number of successes in the jth trial ($0 \leq j \leq n$). Then by Theorem 1.4 (extended to n random variables), and by Theorem 2.1,

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_n) = n \cdot p = np;$$

$$\text{var}(X) = \text{var}(X_1) + \text{var}(X_2) + \cdots + \text{var}(X_n) = n \cdot p(1-p) = np(1-p);$$

and $\text{std}(X) = \sqrt{\text{var}(X)} = \sqrt{np(1-p)}$. □

Example 2.6. Returning to the context of Example 2.4, we compute that

$$E(X) = 4 \cdot 0.8 = 3.2$$

(again),

$$\text{var}(X) = 4 \cdot 0.8 \cdot 0.2 = 0.64$$

(again), and $\text{std}(X) = \sqrt{0.64} = 0.8$ (again).

Section 2 Exercises.

Exercise 2.1. A 40% three-point field goal shooter attempts five three-point shots in a game.

(a) Find the probability mass function for the number X of three-point shots made out of the five. Confirm that your probabilities add up to one.

(b) Find the probability that the player makes at least two three-point shots, out of the five taken. Hint: it might be easier to first compute the probability that they make fewer than two.

(c) Find the expected number of three-point shots made. Please compute this in two ways: (i) using the method of Example 2.4 above (that is, using the definition of expected value, and the probabilities that you computed in part (a) of this exercise), and (ii) using Theorem 2.5. Of course, you should get the same answer either way.

(d) Use Theorem 2.5 to find $\text{var}(X)$ and $\text{std}(X)$.

Exercise 2.2. Consider the following game. You pay \$10, and pick a number from 1 through 6. A fair die is rolled three times. You are awarded \$0 if your number does not come up at all in the three rolls; you are awarded \$20 if your number comes up once, \$30 if it comes up twice, and \$40 if it comes up all three times.

Let X be the number of times your chosen number comes up. Then X is the number of successes in three trials of a binomial experiment, with $P(\text{success}) = 1/6$.

(a) Find the probability mass function for X.

(b) Should you play the game? Hint: your expected payoff, in dollars, is

$$-10 \cdot P(X=0) + 10 \cdot P(X=1) + 20 \cdot P(X=2) + 30 \cdot P(X=3).$$

3 The normal approximation to the binomial distribution

Suppose we have a binomial experiment with $P(\text{success}) = 0.75$, for example, and we want to know the probability of obtaining exactly 70 successes in 100 trials. We can do this by Theorem 2.3:

$$P(X = 70) = \binom{100}{70} \cdot (0.75)^{70} \cdot (0.25)^{30} \approx 0.046.$$

But now, suppose we wanted to compute $P(X \leq 70)$ instead of $P(X = 70)$. At the moment, the only way we have of doing this is to add up a whole bunch of numbers:

$$P(X \leq 70) = P(X = 0) + P(X = 1) + \cdots + P(X = 70).$$

That's pretty tedious. We could make it somewhat less tedious by noting that $P(X \leq 70) = 1 - P(X > 70)$: now we have only 30 probabilities to compute and to add together (namely, the probabilities $P(X = 71)$ through $P(X = 100)$). And of course, either approach is much less tedious if we write a short program to do the calculations by computer. Still, it would be nice if there were a simpler approach.

And there is, provided we don't mind a little approximation. The idea is this: by the Central Limit Theorem, many distributions become approximately normal, provided those distributions are made up of many small, independent factors, each of which behaves similarly. (See Rule 10.3, page 319, of our course text.) In the context of the binomial distribution, the "many small, independent factors" are the n trials of our binomial experiment. Again, we are assuming that our trials are all independent. And we are assuming that $P(\text{success}) = p$ on each trial, so that the n trials do, indeed, behave similarly to each other.

Before stating a concrete result, let's consider the figure below. The histogram depicts $P(X = x)$, for X between 55 and 95, where X is the number of successes in 100 trials of a binomial experiment with $P(\text{success}) = 0.75$. (For $x < 55$ or $x > 95$, the probability of having exactly x successes in this situation is negligible, so we omit these values from the graph.) Superimposed on this histogram is a certain normal curve.

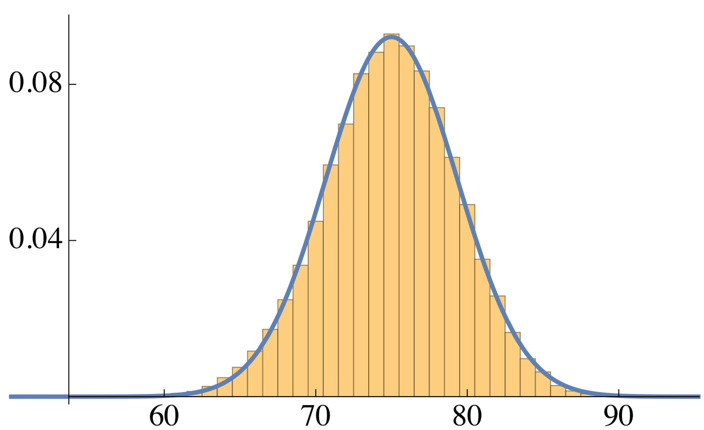

Figure 1. A binomial histogram and a normal pdf

Note that the normal curve fits the graph fairly well. But which normal curve is shown here? It's the normal curve with the same mean – namely, $\mu = np = 100 \cdot 0.75$ – and the same standard deviation – namely, $\sigma = \sqrt{np(1-p)} = \sqrt{100 \cdot 0.75 \cdot 0.25} \approx 4.33$ – as the random variable X.

The above observations are encapsulated by the above theorem.

Theorem 3.1. Suppose X is the number of successes in n independent trials of a binomial experiment with $P(\text{success}) = p$. Suppose also that $np \geq 10$ and $n(1-p) \geq 10$.

Let Y denote $N(np, \sqrt{np(1-p)})$. That is, Y is the normal random variable with the same mean and standard deviation as our binomial random variable X. Then for any numbers j and k between 0 and n inclusive:

(a) $P(X \leq j) \approx P(Y < j + 0.5)$.

(b) $P(X \geq k) \approx P(Y > k - 0.5)$.

(c) $P(k \leq X \leq j) \approx P(k - 0.5 < Y < j + 0.5)$.

Remark: note that, in the above theorem, the normal curve values that are applied in each approximation extend 0.5 units to the left or the right of the corresponding binomial values in question. The reason for this is as follows. Each bar of a histogram for a binomial random variable extends a bit to the left and to the right of the value(s) it represents. For example, we see in Figure 1 above that the bar representing $X = 80$ extends across the interval $[79.5, 80.5]$. Therefore, when approximating binomial probabilities with normal ones, the normal probability density function needs to be considered over an interval large enough to "capture" the entire bar on either end.

An extra half unit of this kind is called a *continuity correction factor*.

To apply the above theorem, one needs to compute normal probabilities. Sometimes these can be looked up, or computed on a calculator, directly, or by first converting to $N(0, 1)$.

Example 3.2. For 100 trials of the binomial experiment described above, with $P(\text{success}) = 0.75$, approximate

(a) $P(X \leq 70)$,

(b) $P(X \geq 65)$;

(c) $P(72 \leq X \leq 78)$.

You may use the facts that, if Z is **standard normal**, then, to four decimal places,

$$P(Z < -1.039) = 0.1492, \qquad P(Z > -2.425) = 0.9923, \qquad P(-0.808 < Z < 0.808) = 0.5809.$$

SOLUTION: Note that $np = 75 \geq 10$ and $n(1-p) = 25 \geq 10$, so the theorem applies.

Let Y be $N(np, \sqrt{np(1-p)}) = N(75, 4.33)$.

(a) By the theorem, $P(X \leq 70) \approx P(Y < 70.5)$. We can compute the latter probability by converting to standard normal, and then using the standard normal probabilities given above. That is,

$$P(Y < 70.5) = P\left(\frac{Y-75}{4.33} < \frac{70.5-75}{4.33}\right) = P\left(\frac{Y-75}{4.33} < -1.039\right) = 0.1492 = 14.92\%,$$

by the given formula $P(Z < -1.039) = 0.1492$.

(b) By the theorem, $P(X \geq 65) \approx P(Y > 64.5)$. As before, we can compute the latter probability by converting to standard normal, and then using the standard normal probabilities given above. That is,

$$P(Y > 64.5) = P\left(\frac{Y-75}{4.33} > \frac{64.5-75}{4.33}\right) = P\left(\frac{Y-75}{4.33} > -2.425\right) = 0.9923 = 99.23\%,$$

by the given formula $P(Z > -2.425) = 0.9923$.

(c) By the theorem, $P(72 \leq X \leq 78) \approx P(71.5 < Y < 78.5)$. Again, we can compute the latter probability by converting to standard normal, and then using the standard normal probabilities given above. That is,

$$\begin{aligned} P(71.5 < Y < 78.5) &= P\left(\frac{71.5-75}{4.33} < \frac{Y-75}{4.33} < \frac{78.5-75}{4.33}\right) \\ &= P\left(-0.808 < \frac{Y-75}{4.33} < 0.808\right) = 0.5810 = 58.09\%, \end{aligned}$$

by the given formula $P(-0.808 < Z < 0.808) = 0.5809$.

Remark: the above probabilities can also be computed directly. For example, as suggested above, we could compute $P(X \leq 70)$ by calculating $P(X = 0), P(X = 1), \ldots, P(X = 70)$ (using Theorem 2.3), and then adding all of these numbers up. Doing so with the help of Mathematica gives

$$P(X \leq 70) = 0.1495,$$

so that our normal approximation of 0.1492, obtained in Example 3.2, is correct to three decimal places.

Similarly, using Mathematica and Theorem 2.3, we compute that

$$P(X \geq 65) = 0.9906, \qquad P(72 \leq X \leq 78) = 0.5811,$$

agreeing with Example 3.2 to at least two decimal places.

Section 3 Exercises.

Exercise 3.1. A basketball player hits 60% of their shots. Suppose all of their shots are independent of each other. Suppose this player takes 25 shots in a game. Let X be the number of shots in this game, out of 25, that the player makes.

(a) Using Theorem 2.3 (that is, *do not* use the normal approximation), compute $P(X = 14)$, $P(X = 15)$, and $P(X = 16)$ directly.

(b) Use your results from part (a) of this exercise to compute $P(14 \leq X \leq 16)$.

(c) In the current setting, are the hypotheses of Theorem 3.1 above satisfied?

(d) Use Theorem 3.1 to approximate $P(14 \leq X \leq 16)$. Use the method of Example 3.2 above. (That is, convert your normal probability to standard normal.) **You may use the fact that, if Z is standard normal, then**

$$P(-0.612 < Z < 0.612) = 0.4597.$$

How do your results compare to those of part (b) of this exercise, above?

Exercise 3.2. A fair die is rolled 180 times. Let X be the number of times a 6 comes up.

(a) Use Theorem 3.1 above to approximate $P(X \leq 35)$. **You may use the fact that, if Z is standard normal, then**

$$P(Z < 1.1) = 0.8643.$$

(b) Use your answer to part (a) of this exercise to approximate $P(X > 35)$.

(c) Now consider the following game: with the above 180 rolls of a fair die, you win \$5 if a 6 comes up at most 35 times, and lose \$30 if a 6 comes up more than 35 times. What are your expected winnings?