Some Poisson Distribution Review Problems

Recall the following. Suppose a certain event happens, on average, λ times per interval of a given extent. Let X be the number of times that the event *actually* happens in such an interval. Then we say "X is Poisson of parameter λ," and we have the formula

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad (k = 0, 1, 2, 3, \ldots). \tag{$*$}$$

1. A certain grocery store closes at 7 PM. If there are more than 5 but at most 10 people waiting at checkout at 6:50 PM, they will open another checkout lane. If there are 10 or more, they will open two additional checkout lanes.

   Suppose, on average, each day there are 4 people waiting at checkout at 6:50 PM.

   (a) Find the probability that, on a given day, the store will need to open exactly one additional lane.

   (b) Find the probability that, on a given day, they will need to open two.

   (c) Find the expected number of additional lanes that the store will need to open on a given day.

2. A certain event happens, on average, 5 times per second.

   (a) Find the probability that this event happens 7 times in a given second.

   (b) Find the probability that it happens 14 times in a given two second interval.

   (c) True or false: the event is roughly half as likely to happen 14 times in two seconds as it is to happen 7 times in one second.

3. Consider a Poisson distribution of parameter $\lambda = n$, where n is a positive integer.

   Show that, for a certain integer k, $P(X = k) = P(X = k + 1)$. Hint: use formula $(*)$ above to write down formulas for $P(X = k)$ and $P(X = k+1)$. Set them equal and solve for k (in terms of n).

4. Show that the formula $(*)$ for the Poisson distribution really does define a probability distribution. This means: show that all the probabilities add up to one, or in other words,

$$\sum_{k=0}^{\infty} P(X = k) = 1.$$

   Hint: you may want to use the Taylor series for e^λ:

$$e^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}.$$

5. In discussing the Poisson distribution in class on Monday, we first came up with the formula

$$P(X = k) = \lim_{N\to\infty} \binom{N}{k} \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k}, \qquad (**)$$

and then showed that, at least for $k = 0, 1$, 2, or 3, **(**)** yields **(*)**. The purpose of this exercise is to show that this holds *any* integer $k = 0, 1, 2, \ldots$.

(a) Use the definition $\binom{N}{k} = \frac{N!}{k!\,(N-k)!}$, together with **(**)**, to show that

$$P(X = k) = \frac{\lambda^k}{k!} \lim_{N\to\infty} \frac{N!}{N^k\,(N-k)!} \left(1 - \frac{\lambda}{N}\right)^{N-k}.$$

(b) Use your answer to the previous part of this problem to find a simple formula (with no limits in it) for $P(X = k)$. Hint: use the following limit formulas (which may be proved using calculus ideas, like l'Hôpital's rule):

$$\lim_{N\to\infty} \frac{N!}{N^k\,(N-k)!} = 1; \quad \lim_{N\to\infty} \left(1 - \frac{\lambda}{N}\right)^N = e^{-\lambda}, \quad \lim_{N\to\infty} \left(1 - \frac{\lambda}{N}\right)^{-k} = 1.$$

Another hint: you know what your answer should be, by **(*)**.