1. Problem 9.10, page 275 in the text: In a charity lottery, one thousand tickets numbered as 0, 001, . . . , 999 are sold. Each contestant buys only one ticket. The prize winners of the lottery are determined by drawing one number at random from the numbers 000, 001, .., 999. You are a prize winner when the number on your ticket is the same as the number drawn or is a random permutation of the number drawn. What is the probability mass function of the number of prize winners and what is the expected value of the number of prize winners? What is the probability that a randomly picked contestant will be a prize winner?

   Let X be the number of prize winners. The possible values of X are 1, 2, and 3 (someone must win because all numbers were sold). Note:

   (a) If all three digits of the number drawn are the same, as in 444, then there is only one winner, since reordering the digits of 444 (for example) still gives you 444.

   (b) If exactly two digits are the same, as in 177, then there are three winners, in this case the person who bought the 177 ticket, the person who bought the 717 ticket, and the person who bought the 771 ticket.

   (c) If all three digits are different, as in 216, then there are six winners, corresponding in this case to the tickets 216, 261, 126, 162, 621, 612.

   There are only ten numbers from 000 through 999 in which all three digits are the same, so $P(X = 1) = \frac{10}{1000} = 0.01$.

   There are 270 numbers from 000 through 999 in which exactly two digits are the same. Why? Say the two equal digits are equal to a, and the other, distinct digit equals b. There are 10 ways of choosing a, leaving 9 ways of choosing b. But then there are $\binom{3}{2} = 3$ ways of choosing which two of the three digits equals a, leaving only one of the three remaining digits that can equal b. And $10 \cdot 9 \cdot 3 = 270$. So $P(X = 3) = \frac{270}{1000} = 0.27$.

   Since there are only three possibilities, we must have $P(X = 6) = 1 - 0.01 - 0.27 = 0.72$.

   So

   $$E(X) = 1 \cdot P(X = 1) + 3 \cdot P(X = 3) + 6 \cdot P(X = 6) = 1 \cdot 0.01 + 3 \cdot 0.27 + 6 \cdot 0.72 = 5.14.$$

   Finally, if a randomly picked contestant is a prize winner, then either there was one winner, there were three winners, or there were six winners. So

   $$\begin{aligned}
   &P(\text{randomly chosen contestant is a winner}) \\
   &= P(\text{one winner}) \cdot P(\text{chosen contestant is a winner given there was only one}) \\
   &+ P(\text{three winners}) \cdot P(\text{chosen contestant is a winner given there were three}) \\
   &+ P(\text{six winners}) \cdot P(\text{chosen contestant is a winner given there were six}) \\
   &= 0.01 \cdot \frac{1}{1000} + 0.27 \cdot \frac{3}{1000} + 0.72 \cdot \frac{6}{1000} = 0.00514.
   \end{aligned}$$

2. A certain basketball player takes 30 shots in a game. Suppose the player is a 60% shooter (meaning 60% of their shots go in). What is the expected number of times that this player will hit two shots in a row, in such a game? Hint: let X_1 equal 1 if the first two shots go in, and 0 otherwise. Let X_2 equal 1 if the second and third shots go in, and 0 otherwise. And so on. Then let $X = X_1 + X_2 + ...$, and compute $E(X)$.

   As suggested, we let X_1 equal 1 if the first two shots go in, and 0 otherwise, let X_2 equal 1 if the second and third shots go in, and 0 otherwise, and so on, all the way up to X_{29}. Note that, for each integer X_j from 1 to 29, we have

   $$E(X_j) = 1 \cdot 0.6 \cdot 0.6 + 0 \cdot 0.64 = 0.36,$$

   since the probability of both shots going in is $0.6 \cdot 0.6$ (and therefore, the probability of less than two of them going in is $1 - 0.6 \cdot 0.6 = 0.64$). Then the s number of times that this player will hit two shots in a row is given by

   $$X = X_1 + X_2 + ... + X_{29}.$$

   So

   $$\begin{aligned} E(X) &= E(X_1 + X_2 + ... + X_{29}) = E(X_1) + E(X_2) + ... + E(X_{29}) \\ &= \underbrace{0.36 + 0.36 + \cdots + 0.36}_{29 \text{ times}} \\ &= 10.44. \end{aligned}$$

3. You pay \$6 to play a game where a fair die is rolled. You lose if the die lands on an even number, you receive \$9 if the die lands on a 1 or a 3, and you receive \$12 if it lands on a 5.

   (a) Find the probability mass function for your payoff X (meaning how much you receive minus the \$6 put in to play).

   The possible values of the payoff X, in dollars, are -6 (if you lose), $9-6 = 3$ (if you roll a 1 or a 3), and $12-6 = 6$ (if you roll a 5). We have $P(X = -6) = P(\text{roll an even number}) = 1/2$, $P(X = 3) = P(\text{roll a 1 or a 3}) = 1/3$, and $P(X = 6) = P(\text{roll a 5}) = 1/6$.

   (b) What are your expected winnings (meaning how much you receive minus the \$6 put in to play) from this game?

   $$E(X) = -6 \cdot P(X = -6) + 3 \cdot P(X = 3) + 6 \cdot P(X = 6) = -6 \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} + 6 \cdot \frac{1}{6} = -1$$

   dollars.

4. You pay \$5 to play the following game. You toss an unfair coin, with P(heads)= 1/3. If the coin lands heads, you choose two marbles at random from a jar containing 3 red marbles and 2 blue marbles. If the coin lands tails, you choose two marbles from a jar containing 4 red marbles and 2 blue marbles. You are then awarded \$15 if you end up with two red marbles; otherwise, you receive \$0.

(a) Find the probability mass function for your payoff X (meaning how much you receive minus the \$5 put in to play).

The payoff X is either -5 dollars or 10 dollars. We have

$$\begin{aligned} P(X=10) &= P(\text{two red marbles}) \\ &= P(\text{coin lands heads}) \cdot P(\text{two red marbles given coin lands heads}) \\ &+ P(\text{coin lands tails}) \cdot P(\text{two red marbles given coin lands tails}) \\ &= \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{2}{4} + \frac{2}{3} \cdot \frac{4}{6} \cdot \frac{3}{5} = \frac{11}{30}. \end{aligned}$$

Since $X = -5$ and $X = 10$ are the only two posibilities, we must have

$$P(X=-5) = 1 - P(X=10) = 1 - \frac{11}{30} = \frac{19}{30}.$$

(b) What are your expected winnings (meaning how much you receive minus the \$5 put in to play) from this game?

$$E(X) = -5 \cdot P(X=-5) + 10P(X=10) = -5 \cdot \frac{19}{30} + 10 \cdot \frac{11}{30} = \frac{-95+110}{30} = \frac{15}{30} = 0.5$$

dollars.

5. Show that, for μ a real number and σ a positive number, the function

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

is a pdf, by showing that

$$\int_{-\infty}^{\infty} f_{\mu,\sigma}(x)\, dx = 1.$$

You may use the fact, which was shown in a homework exercise, that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}.$$

Hint: in the integral you're trying to evaluate, make the substitution $u = (x-\mu)/\sigma$.

If we let $u = (x-\mu)/\sigma$, then $e^{-(x-\mu)^2/(2\sigma^2)} = e^{-u^2/2}$, and $du = dx/\sigma$, so $dx = \sigma\, du$. Also, when $x = -\infty, u = (-\infty - \mu)/\sigma = -\infty$, and when $x = \infty, u = (\infty - \mu)/\sigma = \infty$. So

$$\begin{aligned} \int_{-\infty}^{\infty} f_{\mu,\sigma}(x)\, dx &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\, dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2}\, (\sigma du) = \frac{\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2}\, du \\ &= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = 1. \end{aligned}$$

The next-to-last step is because, as noted above,

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

(and therefore, of course,

$$\int_{-\infty}^{\infty} e^{-u^2/2}\, du = \sqrt{2\pi}$$

as well, since the area doesn't change if you change the name of the variable on the horizontal axis).