Homework Assignment #7: Due Friday, November 10

In this assignment, we derive some properties of the gamma function. (Some of these are given, without proof, in Apostol Section 12.2.)

1. Prove that $\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = \int_0^1 u^{x-1}(1-u)^{y-1}\,du$, for $\mathrm{Re}(x), \mathrm{Re}(y) > 0$. Some hints:

   - Multiply what's on the right by $\Gamma(x+y)$.
   - Make the change of variables $u = \cos^2\theta$, and also change the usual variable of integration t, used in defining the gamma function, to r^2, say.
   - Now think: polar coordinates!

   Try to make some arguments about convergence etc., but don't get hung up on it.

   Remark: the integral on the right above is known as "Euler's beta function" $\mathrm{B}(x,y)$; so $\mathrm{B}(x,y) = \Gamma(x)\,\Gamma(y)/\Gamma(x+y)$ for $\mathrm{Re}(x), \mathrm{Re}(y) > 0$.

2. Use the previous exercise to evaluate $\Gamma(1/2)$.

3. Prove that, for $\mathrm{Re}(s) > 0$, $\Gamma(s)\Gamma(s+1/2) = \sqrt{\pi}2^{1-2s}\Gamma(2s)$ (this is the so–called duplication formula for the gamma function). Hint: use part (a) above, and a change of variable, to show $\mathrm{B}(1/2, s) = \int_{-1}^{1}(1-v^2)^{s-1}\,dv$. Now put $v = 2w - 1$.

4. For this exercise you should recall that, for functions $f, g : \mathbb{R} \to \mathbb{C}$, we define the *convolution* $f * g$ of f and g to be the function on $\mathbb{R}$ defined by

   $$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy,$$

   for all x such the integral exists. Also, we say that a set S of functions on $\mathbb{R}$ is a *convolution semigroup* if $f * g \in S$ whenever $f, g \in S$. Show that, if we define a function f_p on $\mathbb{R}$, for each complex number p with $\mathrm{Re}(p) > 0$, by

   $$f_p(x) = \begin{cases} 0 & \text{if } x \le 0, \\ \dfrac{x^{p-1}e^{-x}}{\Gamma(p)} & \text{if } x > 0, \end{cases}$$

   then the set

   $$S = \{f_p \colon p \in \mathbb{C},\ \mathrm{Re}(p) > 0\}$$

   is a convolution semigroup.