$L(1,\chi) \neq 0$, part A: the case of real χ.

Theorem 6.19.
If χ is real and
$$A(n) = \sum_{d \mid n} \chi(d),$$
then $A(n) \geq 0$ for all n. Moreover, if n is a square, then $A(n) \geq 1$ for all n.

Proof.
$A(1) = \chi(1) = 1$; the case $n=1$ follows.

Now suppose
$$n = \prod_{i=1}^{r} p_i^{a_i}$$
with each p_i prime and each $a_i \in \mathbb{Z}_+$. Since $A = \chi * u$ and χ and u are multiplicative, so is A, so
$$A(n) = \prod_{i=1}^{r} A(p_i^{a_i}).$$

Note also that n is a square iff each a_i is even. So it's enough to show that, for p prime and $a \in \mathbb{Z}_+$, (i) $A(p^a) \geq 0$, and (b) $A(p^a) \geq 1$ for a even.

For such p and a, we have
$$A(p^a) = \sum_{d \mid p^a} \chi(d) = \sum_{t=0}^{a} \chi(p^t) = \sum_{t=0}^{a} \chi(p)^t.$$

Now since χ is real, we have only three possibilities: $\chi(p) = 1$, $\chi(p) = 0$, or $\chi(p) = -1$.

If $\chi(p)=1$, the sum on the right equals $1+a$, which is ≥ 1, and we're done. If not, this sum equals

$$\frac{1-\chi(p)^{a+1}}{1-\chi(p)} = \begin{cases} 1 & \text{if } \chi(p)=0; \\ 0 & \text{if } \chi(p)=-1 \text{ and } a \text{ is odd}; \\ 1 & \text{if } \chi(p)=-1 \text{ and } a \text{ is even,} \end{cases}$$

and we're done. □

Next, we need a lemma:

<u>Theorem 3.17.</u>
Let f, g be arithmetic functions; let

$$F(x) = \sum_{n \leq x} f(n), \qquad G(x) = \sum_{n \leq x} g(n).$$

Then, for any $a, b \in \mathbb{R}^+$ with $ab = x$, we have

$$\sum_{n \leq x} f*g(n) = \sum_{n \leq a} f(n)G(x/n) + \sum_{n \leq b} g(n)F(x/n) - F(a)G(b).$$

[Note that putting $(a,b)=(1,x)$ and $(a,b)=(x,1)$ into Theorem 3.17 gives Thm. 3.10.]

<u>Proof.</u>
We have

$$\sum_{n \leq x} f*g(n) = \sum_{n \leq x} \sum_{d \mid n} f(d)g(n/d)$$
$$= \sum_{cd \leq x} f(d)g(c),$$

by putting $c = n/d$. Now note that, if $ab = x$, then

$$\{(c,d) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : cd \le x\}$$

$$= \{(c,d) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : cd \le x \text{ and } d \le a\}$$
$$\cup \{(c,d) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : cd \le x \text{ and } c \le b\}.$$

The union is not disjoint: the intersection of the sets on the right is

$$\{(c,d) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : cd \le x,\ d \le a, \text{ and } c \le b\}.$$

So

$$\sum_{n \le x} f * \underset{\sim}{g}(n) = \sum_{cd \le x} f(d)\underset{\sim}{g}(c)$$

$$= \sum_{\substack{cd \le x \\ d \le a}} f(d)\underset{\sim}{g}(c) + \sum_{\substack{cd \le x \\ c \le b}} f(d)\underset{\sim}{g}(c) - \sum_{\substack{cd \le x \\ d \le a \\ c \le b}} f(d)\underset{\sim}{g}(c)$$

$$= \sum_{d \le a} f(d) \sum_{c \le x/d} \underset{\sim}{g}(c) + \sum_{c \le b} \underset{\sim}{g}(c) \sum_{d \le x/c} f(d) - \sum_{d \le a} f(d) \sum_{c \le b} \underset{\sim}{g}(c)$$

$$= \sum_{d \le a} f(d) G(x/d) + \sum_{c \le b} \underset{\sim}{g}(c) F(x/c) - F(a)G(b),$$

as claimed. $\square$

Next, we have

<u>Theorem 6.20.</u>

Let χ be real and nonprincipal; let

$$A(n) = \sum_{d|n} \chi(n) \text{ and } \quad B(x) = \sum_{n \le x} \frac{A(n)}{\sqrt{n}}.$$

Then
(a) $B(x) \to \infty$ and $x \to \infty$, and
(b) $B(x) = \sqrt{x}\, L(1, \chi) + O(1)$

for $x \geq 1$.

Proof.
(a) By Thm. 6.19,

$$B(x) = \sum_{n \le x} \frac{A(n)}{\sqrt{n}} \geq \sum_{\substack{n \le x \\ n \text{ is a square}}} \frac{A(n)}{\sqrt{n}} = \sum_{\substack{m \in \mathbb{Z}_+ \\ m^2 \le x}} \frac{A(m^2)}{m}$$

$$= \sum_{m \le \sqrt{x}} \frac{A(m^2)}{m} \geq \sum_{m \le \sqrt{x}} \frac{1}{m},$$

which $\to \infty$ as $x \to \infty$ because the harmonic series diverges.

(b) We have

$$B(x) = \sum_{n \le x} \frac{1}{\sqrt{n}} \sum_{d|n} \chi(d) = \sum_{n \le x} \sum_{d|n} \frac{\chi(d)}{\sqrt{d}} \cdot \frac{1}{\sqrt{n/d}}$$

$$= \sum_{n \le x} f * g(n),$$

where $f(n) = \chi(n)/\sqrt{n}$ and $g(n) = 1/\sqrt{n}$.

If we define

$$F(x) = \sum_{n \le x} f(n) = \sum_{n \le x} \frac{\chi(n)}{\sqrt{n}} \text{ and } G(x) = \sum_{n \le x} g(n) = \sum_{n \le x} \frac{1}{\sqrt{n}},$$

then by Thm. 6.19 with $a = b = \sqrt{x}$,

$$B(x) = \sum_{n \le \sqrt{x}} f(n) G(x/n) + \sum_{n \le \sqrt{x}} g(n) F(x/n) - F(\sqrt{x}) G(\sqrt{x}).$$

We'll finish this proof next time. In the meantime note that, if $L(1,\chi) = 0$, then Thm. 6.20(b) implies $B(x) = O(1)$, contradicting Thm. 6.20(a). So assuming Thm. 6.20(b), we have

<u>Corollary</u>

χ real, nonprincipal $\Rightarrow L(1,\chi) \neq 0$.