MATH 3510: Prob/Stats

November 15, 2023

Second In-class Midterm Exam (SOLUTIONS)

1. (32 points; 8 points each)

You select two marbles, one after the other, at random from a jar that contains four red marbles and two blue marbles. (You choose the marbles without replacement; that is, you leave the first marble out of the jar when selecting the second marble.) Let X denote the number of red marbles that you end up with.

Note: you might want to check your work below by checking that the relevant probabilities add up to one. Also, it **might** be easier to do parts (a) and (c) of this problem first, and then come back and use that info to do part (b).

(a) Find $P(X = 0)$. Solution:

$$P(X = 0) = P(\text{both blue}) = \frac{2}{6} \cdot \frac{1}{5} = \frac{2}{30} = \frac{1}{15}.$$

(b) Find $P(X = 1)$. Solution:

$$\begin{aligned} P(X = 1) &= P(\text{one red}) \\ &= P(\text{the first is red and the second is blue}) \\ &\quad + P(\text{the first is blue and the second is red}) \\ &= \frac{4}{6} \cdot \frac{2}{5} + \frac{2}{6} \cdot \frac{4}{5} = \frac{8}{30} + \frac{8}{30} = \frac{16}{30} = \frac{8}{15}. \end{aligned}$$

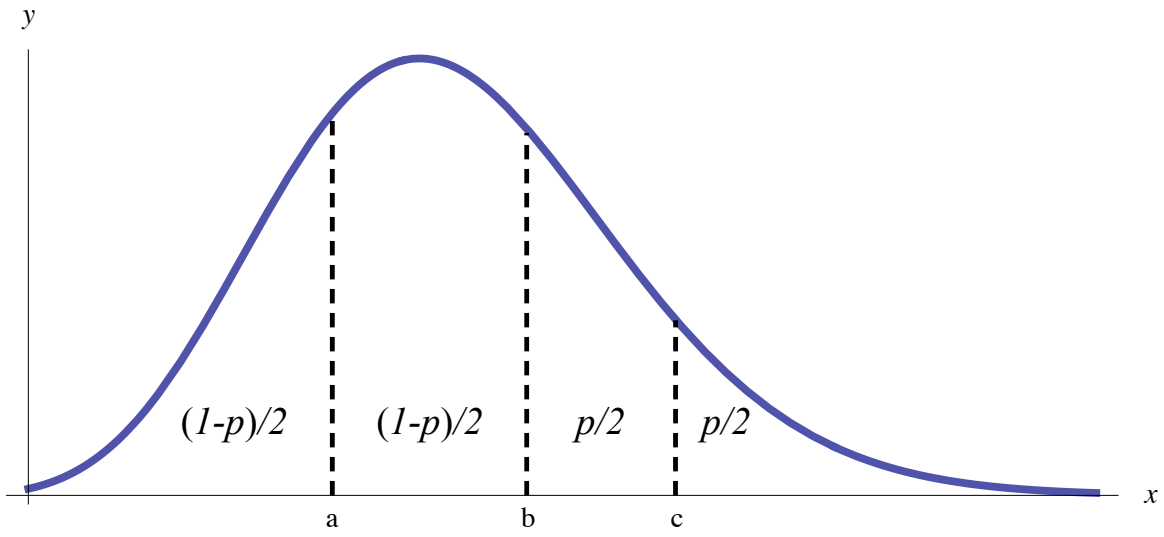
(c) Find $P(X = 2)$. Solution:

$$P(X = 2) = P(\text{both red}) = \frac{4}{6} \cdot \frac{3}{5} = \frac{12}{30} = \frac{6}{15}.$$

(d) Find $E(X)$. Solution:

$$\begin{aligned} E(X) &= 0 \cdot P(X = 0) + 1 \cdot P(X = 1) + 2 \cdot P(X = 2) \\ &= 0 \cdot \frac{1}{15} + 1 \cdot \frac{8}{15} + 2 \cdot \frac{6}{15} = \frac{8 + 12}{15} = \frac{20}{15} = \frac{4}{3}. \end{aligned}$$

2. (32 points; 8 points each) The following graph corresponds to the probability density function for a random variable X. The regions delineated by dashed lines have areas as shown – here, p is a constant that's less than 1.



For parts (a)(b)(c) below, your answers should be in terms of p.

(a) Find $P(x < a)$. $P(x < a) = (1 - p)/2$.

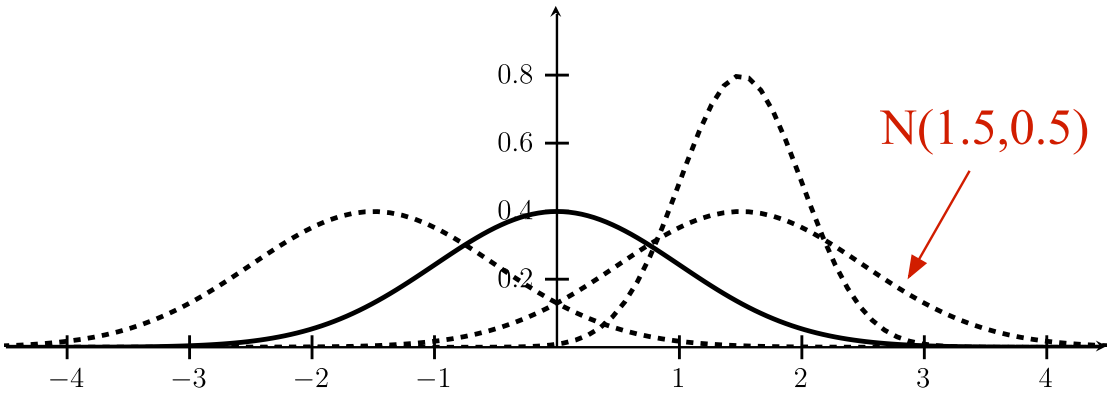
(b) Find $P(x < c)$. $P(x < a) = \frac{1-p}{2} + \frac{1-p}{2} + \frac{p}{2} = \frac{2-p}{2} = 1 - \frac{p}{2}$.

(c) Find the number r such that $P(x > r) = p$. $P(x > b) = \frac{p}{2} + \frac{p}{2} = p$, so $r = b$.

(d) Looking at the graph, about what do you think p is equal to? (Your answer should be a real number.) Please explain. Note: there are many plausible answers here. Any plausible answer will be accepted, provided your explanation backs it up.

The total area under a pdf is 1. It looks like the area to the right of the vertical line $x = c$ has about 1/6 the total area, so it looks like, roughly, $p/2 = 1/6$, or $p = 1/3$.

3. (15 points; 5 points each) The solid black line featured below is a $N(0,1)$ pdf, and the dotted lines are other normal curves.



(a) Please mark which curve corresponds to a $N(1.5, 0.5)$ pdf. **Please explain how you know.** The indicated curve is the $N(1.5, 0.5)$ curve. I know this because $N(1.5, 0.5)$ denotes a normal curve with mean 1.5 and standard deviation 0.5. There are only two curves with mean 1.5. The shorter of these two curves has the same spread, and therefore the same standard deviation, as the $N(0, 1)$ curve, so it can't be $N(1.5, 0.5)$. So the taller, and less spread out, of the two curves centered at 1.5 must be the $N(1.5, 0.5)$ curve.

(b) Suppose that X is $N(1.5, 0.5)$. Find

$$P(0 < X < 3).$$

$$P(0 < X < 3) = P\left(\frac{0 - 1.5}{0.5} < \frac{X - 1.5}{0.5} < \frac{3 - 1.5}{0.5}\right)$$
$$= P(-3 < X < 3) = 0.997.$$

(c) Suppose, once more, that X is $N(1.5, 0.5)$. Consider the set

$$\overline{X} = \{\text{sample means from size-100 samples of } X\}.$$

Which of the following equals the standard deviation $\overline{\sigma}$ of $\overline{X}$ (circle the correct answer; you don't need to explain):

1.5 0.15 0.5 0.05 98% 2.576

4. (11 points) While vacationing on Central Limit Island, you start to wonder how much coconut milk is trapped inside the average coconut. You take a random sample of 49 island coconuts and find that, on average in your sample, there are $\overline{x} = 9.05$ ounces of coconut milk in each coconut, with a sample standard deviation of $s = 1.55$ ounces.

Create a 99% confidence interval for the true (population) mean number of ounces of coconut milk in all coconuts on the island.

$$\left(9.05 - 2.576 \cdot \frac{1.55}{\sqrt{49}}, 9.05 + 2.576 \cdot \frac{1.55}{\sqrt{49}}\right) = (8.48, 9.62).$$

5. (10 points)

(All numbers and scenarios here are as in the previous exercise.) The Coconut Adjudication Bureau is also vacationing on Central Limit Island, and they claim that the true mean number of fluid ounces of coconut milk is 10 ounces per coconut. At the 99% level, do you reject this claim, or do you not reject it? Please explain. Note: you may do this problem by conducting a hypothesis test, using all data from the previous exercise, or by referring to the confidence interval of the previous exercise. If you use the latter method, please explain how and why that confidence interval gives you the answer to this exercise.

I reject this claim at the 99% level, because the number 10 falls *outside* of the 99% confidence interval for the mean number of fluid ounces of coconut milk in a Central Limit Island coconut.