The Poisson distribution.

Recall:

suppose a certain event happens, on average, λ times in each interval of a "certain extent." Let X be the number of times the event actually happens in such an interval.

Then for $k = 0, 1, 2, 3, \ldots$, we have

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}.$$

We say that X has a Poisson distribution of type λ, or simply "X is $P(\lambda)$."

Example 1.

In 1911, Ernest Rutherford et al. observed that, on average (over several hours), a sample of polonium emitted $\lambda = 3.8715$ α-rays every 7.5 seconds.

Let X be the number of rays emitted in a random 7.5 second period. Find $P(X=k)$ for $0 \le k \le 5$.

Solution.

We have

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda} = \frac{3.8715^k e^{-3.8715}}{k!}.$$

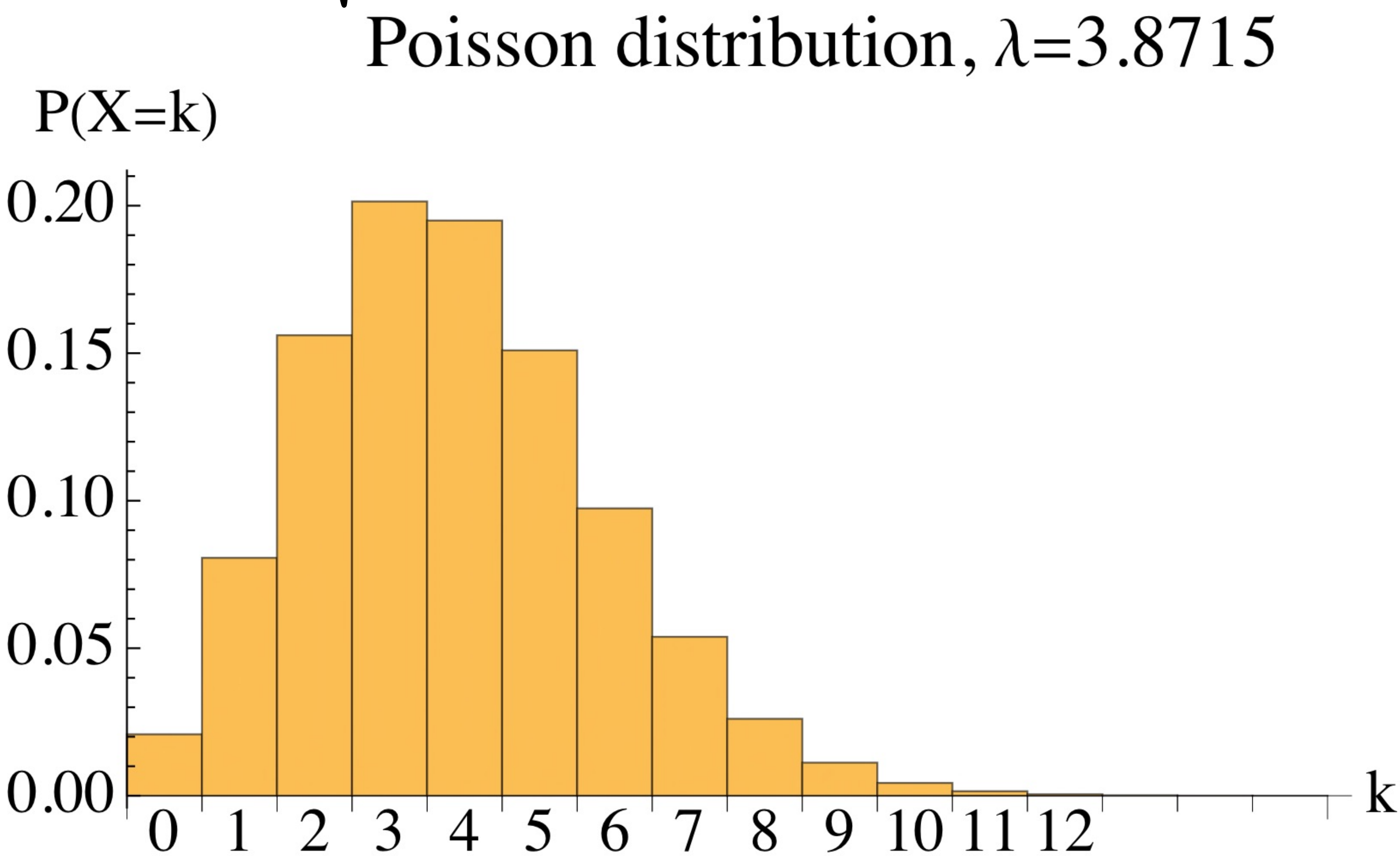
E.g.

$$P(X=0) = \frac{3.8715^{0} e^{-3.8715}}{0!} = e^{-3.8715} = 0.0208.$$

And so on:

k	$P(X=k) = 3.8715^{k} \cdot e^{-3.8715}/k!$
0	0.0208
1	0.0806
2	0.1561
3	0.2014
4	0.1950
5	0.1510

Here's a picture:



Note that the mean appears to be somewhere between 3 and 4. This is because:

FACT.

If X is $P(\lambda)$, then
$$E(X) = \lambda \quad \text{and} \quad var(X) = \lambda.$$

Proof

First we compute $E(X)$:

$$E(X) = \sum_{k=0}^{\infty} k \cdot P(X=k)$$

$$= 0 \cdot e^{-\lambda} + 1 \cdot \lambda e^{-\lambda} + 2 \cdot \frac{\lambda^2}{2!} e^{-\lambda} + 3 \cdot \frac{\lambda^3}{3!} e^{-\lambda} + 4 \cdot \frac{\lambda^4}{4!} e^{-\lambda} + \dots$$

note that $\frac{k}{k!} = \frac{1}{(k-1)!}$

$$= \lambda e^{-\lambda} + \frac{\lambda^2}{1!} e^{-\lambda} + \frac{\lambda^3}{2!} e^{-\lambda} + \frac{\lambda^4}{3!} e^{-\lambda} + \dots$$

factor out $\lambda e^{-\lambda}$

$$= \lambda e^{-\lambda} \underbrace{\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \right)}_{\text{this is the power series for } e^{\lambda}!}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

$var(X)$ is computed similarly. $\square$

Example 2

A CPU receives an average of 3 instructions per nanosecond. It will crash if it receives > 15 instructions in a nanosecond.

What's the probability of a crash?

Solution.

Let X be the number of instructions received in a nanosecond. Then

$$P(\text{crash}) = P(X>15)$$
$$= 1 - P(X \le 15)$$
$$= 1 - \sum_{k=0}^{15} \frac{3^k \cdot e^{-3}}{k!}$$
$$= 1.2408 \times 10^{-7}.$$

Example 3

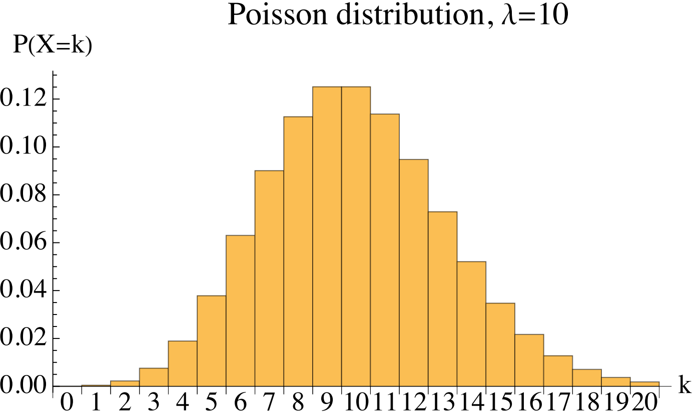
KC Chiefs tight end Travis Kelce drops, on average, 0.25 catchable passes per game. What's the probability that, in a given game, he'll drop more than one such pass?

Solution. If X is the number of catchable passes that TK drops in a game, then X is $P(0.25)$. We compute:

$$P(X=0) = e^{-0.25} = 0.7788,$$
$$P(X=1) = 0.25e^{-0.25} = 0.1947,$$

so

$$P(X>1) = 1 - P(X \le 1)$$
$$= 1 - P(X=0) - P(X=1)$$
$$= 1 - 0.7788 - 0.1947$$
$$= 0.0265 = 2.65\%.$$



Poisson distribution, λ=10
P(X=k)
0.12
0.10
0.08
0.06
0.04
0.02
0.00
k
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20