The Poisson distribution.

Consider an event that happens, <u>on average</u>, λ times in every interval (of time, space, etc.) of some fixed extent.

<u>Examples</u>:

(a) On average, in a meteor shower, 5 meteorites hit the earth every square kilometer.

(b) On average, in Target during peak hours, 25 people enter the self-checkout line every 15 minutes.

(c) On average, a computer CPU, running certain software, receives 3 instructions per nanosecond.

(d) On average, a sample of polonium emits 3.8715 α-rays per 7.5 second period.

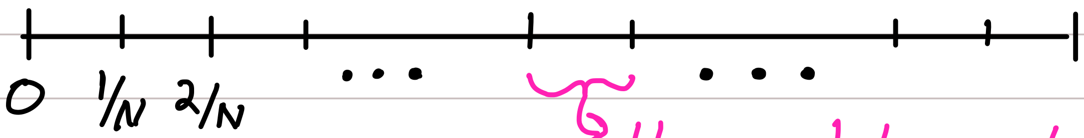
(e) On average, a large computer program contains 4 errors per 10,000 lines of code.

Let X denote the number of occurences of this event in an interval of the given extent. Question: given <u>any</u> integer k, with $k \geq 0$, what is $P(X=k)$?

[Note: unlike the binomial distribution, there's no upper limit n here on how large k can be.]

To answer, let's choose a large integer N, and divide our interval, of the given length, into N evenly spaced subintervals.
Assume N is large enough that the event is <u>very unlikely</u> to happen more than once on any subinterval.

Since our event happens (on average) λ times on the original interval, the probability of the event happening in any of the N given <u>subintervals</u> is about λ/N.



the event happens here with probability λ/N.

In order for the event to happen k times on the original interval, it must:

(i) happen in k of the subintervals; the probability of this is λ/N for <u>each</u> of these subintervals;
(ii) <u>not</u> happen in $N-k$ of the subintervals: the probability of this is $(1-\lambda/N)$ for each of these subintervals.

There are $\binom{N}{k}$ ways of choosing the k subintervals in (i). So, similarly to the binomial distribution,

$$P(X=k) \approx \binom{N}{k}\left(\frac{\lambda}{N}\right)^k\left(1-\frac{\lambda}{N}\right)^{N-k}.$$

But note: N can be as large as we want! So we find that:

For a Poisson distribution with an average of λ events per specified interval I of a specified length,

$$P(X=k) = \lim_{N\to\infty} \binom{N}{k}\left(\frac{\lambda}{N}\right)^k \left(1-\frac{\lambda}{N}\right)^{N-k}$$

$(k=0,1,2,3,\ldots)$

We call λ the <u>parameter</u> of the distribution.

<u>Example 2</u>

Find $P(X=k)$ for a Poisson distribution of parameter λ, when

(a) $k=0$,

(b) $k=1$,

(c) $k=2$.

Use the fact, from calculus, that, if k and λ are fixed, then

$$\lim_{N\to\infty}\left(1-\frac{\lambda}{N}\right)^{N-k} = e^{-\lambda}. \qquad (*)$$

<u>Solution</u>

(a) $$P(X=0) = \lim_{N\to\infty}\binom{N}{0}\left(\frac{\lambda}{N}\right)^0\left(1-\frac{\lambda}{N}\right)^N$$

$$= \lim_{N\to\infty} 1\cdot 1\cdot\left(1-\frac{\lambda}{N}\right)^N = e^{-\lambda},$$

by $(*)$.

(b) $P(X=1) = \lim_{N\to\infty} \binom{N}{1}\left(\frac{\lambda}{N}\right)^1\left(1-\frac{\lambda}{N}\right)^{N-1}$

$= \lim_{N\to\infty} N\cdot\frac{\lambda}{N}\cdot\left(1-\frac{\lambda}{N}\right)^{N-1}.$

$= \lim_{N\to\infty} \lambda\cdot\left(1-\frac{\lambda}{N}\right)^{N-1}$

$\to e^{-\lambda}$ as $N\to\infty$, by $(*)$

$= \lambda e^{-\lambda}.$

(c) $P(X=1) = \lim_{N\to\infty} \binom{N}{2}\left(\frac{\lambda}{N}\right)^2\cdot\left(1-\frac{\lambda}{N}\right)^{N-2}$

$= \lim_{N\to\infty} \frac{N(N-1)}{2}\cdot\frac{\lambda^2}{N^2}\cdot\left(1-\frac{\lambda}{N}\right)^{N-2}$

$= \frac{\lambda^2}{2}\lim_{N\to\infty} \frac{N^2-N}{N^2}\boxed{\left(1-\frac{\lambda}{N}\right)^{N-2}}$

$\to 1$ as $N\to\infty$ $\quad$ $\to e^{-\lambda}$ as $N\to\infty$, by $(*)$

$= \frac{\lambda^2}{2}e^{-\lambda}.$

Q: what's the pattern? Answer:

$P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}.$