More on mean, variance, standard deviation, and binomial distributions.

A) Mean, variance, standard deviation.

Recall: if X is a discrete random variable, then:

$$E(X) = \sum_x x \cdot P(X=x)$$

($E(X)$ is also written $\mu(X)$ or just μ);

$$\mathrm{var}(X) = E((X-\mu)^2) = \sum_x (x-\mu)^2 P(X=x)$$

($\mathrm{var}(X)$ is also written $\sigma^2(X)$ or just σ^2);

$$\mathrm{std}(X) = \sqrt{\mathrm{var}(X)}$$

($\mathrm{std}(X)$ is also written $\sigma(X)$ or just σ).

<u>Example 1</u>

(a) A fair die is rolled. Let X be the number that shows. Find $E(X)$, $\mathrm{var}(X)$, $\mathrm{std}(X)$.

(b) Repeat part (a) if the die is unfair, with $P(1) = 1/3$, $P(2)=P(3)=P(4)=P(5)=P(6)= 2/15$.

<u>Solution.</u>

(a) $E(X) = 1 \cdot P(X=1) + 2 \cdot P(X=2) + \dots + 6 \cdot P(X=6)$
$= 1 \cdot 1/6 + 2 \cdot 1/6 + \dots + 6 \cdot 1/6$
$= 1/6\,(1+2+\dots+6) = 21/6 = 3.5.$

$\mathrm{var}(X) = (1-3.5)^2 \cdot P(X=1) + (2-3.5)^2 \cdot P(X=2)$
$+ \dots + (6-3.5)^2 \cdot P(X=6)$

$= 1/6\left((1-3.5)^2 + \dots + (6-3.5)^2\right)$

$$= 2.917,$$
$$\mathrm{std}(X) = \sqrt{\mathrm{var}(X)} = 1.708.$$

(b) $E(X) = 1 \cdot 1/3 + 2 \cdot 2/15 + 3 \cdot 2/15 + \ldots + 6 \cdot 2/15$
$= 3,$

$$\mathrm{var}(X) = (1-3)^2 \cdot 1/3 + (2-3)^2 \cdot 2/15 + \ldots + (6-3)^2 \cdot 2/15$$
$$= 10/3 = 3.\overline{3}$$

$$\mathrm{std}(X) = \sqrt{10/3} = 1.826.$$

Note: we've seen before that

$$E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n).$$

It's also true that, <u>if</u> all of the X_j's are independent, then

$$\mathrm{var}(X_1 + X_2 + \ldots + X_n) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + \ldots + \mathrm{var}(X_n). \quad (*)$$

<u>Example 2</u>

Suppose the die of Example 1(b) is rolled 10 times. Let X denote the sum of the results of the 10 rolls. Find $E(X)$, $\mathrm{var}(X)$, $\mathrm{std}(X)$.

<u>Solution</u>. Write

$$X = X_1 + X_2 + \ldots + X_{10},$$

where X_j is the number showing on the j^{th} roll. Then

$$E(X) = E(X_1) + E(X_2) + \ldots + E(X_{10})$$
$$= \underbrace{3 + 3 + \quad + 3}_{10 \text{ times}}$$

$= 30$

$$\mathrm{var}(X) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + \ldots + \mathrm{var}(X_{10})$$
$$= 10/3 + 10/3 + \ldots + 10/3$$
$$= 100/3 = 33.\overline{3},$$

$$\mathrm{std}(X) = \sqrt{100/3} = 5.774.$$

B) Binomial distribution.

A binomial experiment is one with only two possible outcomes, a "success" and a "failure." Let's write $P(\text{success}) = p$, so that $P(\text{failure}) = 1-p$.

Let $X=1$ if the experiment is a success and $X=0$ if not. We've seen that:

$$E(X) = p, \quad \mathrm{var}(X) = p(1-p), \quad \mathrm{std}(X) = \sqrt{p(1-p)}.$$

We've also seen that, if X is the number of successes in $\underline{\underline{n}}$ trials of this experiment, then:

(i) $E(X) = np$;

(ii) $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$ $\quad (0 \leq k \leq n)$.

<u>NOW</u>, using $(*)$ above, we find that, in this same situation,

(iii) $\mathrm{var}(X) = np(1-p)$ and $\mathrm{std}(X) = \sqrt{np(1-p)}$.

Example 3

Six fair coins are flipped. Let X be the number of heads showing. Find:

(a) $P(X=1)$ and $P(X=3)$;
(b) $E(X)$, $var(X)$, $std(X)$.

Solution (a) By (ii) above,

$$P(X=1) = \binom{6}{1}(.5)^2(.5)^{6-2} = 0.0938,$$
$$P(X=3) = \binom{6}{3}(.5)^3(.5)^{6-3} = 0.3125.$$

(b) By (i) and (iii) above,

$$E(X) = 6 \cdot 0.5 = 3,$$
$$var(X) = 6 \cdot 0.5 \cdot (1-0.5) = 1.5,$$
$$std(X) = \sqrt{1.5} = 1.2247.$$

Note: a while ago, we flipped six coins and recorded the # of heads X repeatedly (591 times). We found that X equalled one $63/591 = 10.66\%$ of the time, and equalled three $189/591 = 31.98\%$ of the time.

Also, on average, 3.0067 coins landed heads, with a standard deviation of 1.2262.