Brief digression: standard deviation of a random variable. (Section 5.2.1)

Part I: Review of expected value.

First, recall that the expected value of an rv X is defined by

$$E(X) = \sum_{x} x \cdot P(X=x), \qquad (*)$$

where the sum is over all possible values x of X.

Example 1.

Consider a binomial experiment, meaning one with only two possible outcomes, a "success" and a "failure".

Suppose $P(\text{success}) = p$.

Let X be the number of successes in a single trial. That is, $X=1$ if a success occurs, and $X=0$ if a failure occurs.

Then by $(*)$,

$$\begin{aligned} E(X) &= 1 \cdot P(X=1) + 0 \cdot P(X=0) \\ &= 1 \cdot P(\text{success}) + 0 \cdot P(\text{failure}) \\ &= 1 \cdot p + 0 \cdot (1-p) \\ &= p. \end{aligned}$$

This makes intuitive sense: if $P(\text{success}) = p$, then we would expect the number of successes on a single trial to be p (on average).

Example 2.

Now, suppose our binomial experiment, with $P(\text{success}) = p$, is repeated n times. Let X be the number of successes in these n trials. What is $E(X)$?

Solution. Write

$$X = X_1 + X_2 + \dots + X_n,$$

where X_j is the number of successes in the j^{th} trial. Then, by the sum rule for expected value,

$$E(X) = E(X_1) + E(X_2) + \dots + E(X_n)$$
$$= \underbrace{p + p + \dots + p}_{n \text{ times}} = np.$$

Part II. Variance and std dev of an rv.

We define the variance $\text{var}(X)$ of an rv X by

$$\text{var}(X) = E((X-\mu)^2),$$

where $\mu = E(X)$. We also define the std dev $\text{std}(X)$ of X by $\text{std}(X) = \sqrt{\text{var}(X)}$.

Example 3.

Find the variance and std dev of a binomial rv X with $P(\text{success}) = p$.

Solution

We have

$$\text{var}(X) = E((X-\mu)^2) = E((X-p)^2),$$

since we know $\mu = E(X) = p$.

Now X can equal 1 or 0, with probability p and $1-p$ respectively. So $(X-p)^2$ can equal $(1-p)^2$ or $(0-p)^2 = p^2$, with probability p and $1-p$ respectively. So, by (*),

$$\mathrm{var}(X) = E((X-p)^2) = (1-p)^2 \cdot p + p^2 \cdot (1-p)$$

factor out $p(1-p)$

$$= p(1-p)(1-p+p) = p(1-p),$$

and

$$\mathrm{std}(X) = \sqrt{p(1-p)}.$$

Summary

Let X be a binomial rv, meaning X only takes the values 1 or 0, where

$$P(X=1) = p \text{ and } P(X=0) = 1-p.$$

Then X has expected value

$$E(X) = p,$$

variance

$$\mathrm{var}(X) = E((X-\mu)^2) = E((X-p)^2) = p(1-p),$$

and std dev

$$\mathrm{std}(X) = \sqrt{p(1-p)}.$$

Example 4.

If a free throw shooter has an 80% success rate, then the expected number of made shots in a single attempt is

$$E(X) = p = 0.8,$$

with std dev.

$$std(X) = \sqrt{0.8 \cdot 0.2} = 0.4.$$

Next question: if X is the # of successes in n trials of a binomial experiment, what is $std(X)$??