Applications of the SDM theorem II: Confidence intervals.

The SDM theorem tells us that in a population X with mean μ and std. dev. σ, for a fixed $n \geq 30$, the "Z statistic"

$$Z = \frac{\bar{X} - \mu}{(s/\sqrt{n})} \quad \text{is roughly } N(0,1)$$

(under mild conditions).

So for example, in such a case,

$$P\left(-2.576 < \frac{\bar{X} - \mu}{(s/\sqrt{n})} < 2.576\right) = 99\%.$$

Now inside of the P( ), do some algebra to get μ by itself in the middle:

(a) Multiply everything through by $-s/\sqrt{n}$.

(b) Add $\bar{X}$ to each term.

The result is:

$$P\left(\bar{X} - 2.576\frac{s}{\sqrt{n}} < \mu < \bar{X} + 2.576\frac{s}{\sqrt{n}}\right) = 99\%.$$

Interpretation: 99% of the time, when a size-n random sample from X is chosen, and $\bar{x}$ and s are computed, the interval

$$\left(\bar{x} - 2.576\,s/\sqrt{n},\ \bar{x} + 2.576\,s/\sqrt{n}\right) \qquad (CI_{99})$$

will contain the true population mean μ.

The interval (CI_{99}) is called a 99% confidence interval for μ.

<u>Example.</u> In studying the Etruscan Empire (~700–300 BC), anthropologists measure the breadth of a "random" sample of $n = 84$ male Etruscan skulls, and find that

$$\bar{x} = 143.77 \text{ mm}, \quad s = 5.97 \text{ mm}.$$

Construct a 99% confidence interval for mean male Etruscan skull breadth μ.

<u>Solution.</u>

The interval is

$$\left(143.77 - 2.576 \cdot 5.97/\sqrt{84},\ 143.77 + 2.576 \cdot 5.97/\sqrt{48}\right)$$

$$= (142.09, 145.45).$$

<u>Notes.</u>

1) Note that the interval (CI_{99}) is centered about the sample mean $\bar{x}$.

2) A 95% or 98% confidence interval for μ would look like (CI_{99}), but with 2.576 replaced by 1.96 or 2.33 respectively.

3) So: <u>more</u> <u>confidence</u> requires a <u>wider</u> interval.

<u>Example 2.</u>

Let μ be as in Example 1. Test the null hypothesis

$$H_0: \mu = 132.44 \text{ mm}$$

against the alternative hypothesis

$$H_A: \mu \neq 132.44 \text{ mm}$$

at the 99% level.

[Note: 132.44 is the mean skull breadth of present-day Italian males. So this test helps answer whether Etruscans were native to Italy.]

<u>Solution:</u>

We compute the z-statistic

$$z = \frac{\bar{x} - \mu_0}{(s/\sqrt{n})} = \frac{143.77 - 132.44}{(5.97/\sqrt{84})}$$

$$= 17.39.$$

Since $|z| > 2.576$, we reject H_0, and accept H_A, at the 99% level.

Note: in general,

"reject $H_0: \mu = \mu_0$, and accept $H_A: \mu \neq \mu_0$, at the p % level"

is the same as

"μ_0 is <u>outside</u> the p % confidence interval for μ."