GOAL: To use Spirographs to model, and solve, *linear congruences*. What does that mean? It means: *given* integers a, b, and c, we want to *find* integers x and y such that

$$ax - by = c.$$

NOTE: linear congruences have lots of applications in math. They also have applications in computer science: for example, in cryptography, error detection, etc. If you're interested, see the article

> Senad Orhani and Besim Çeko, Some Applications of linear conguences from number theory, *Intenational Research Journal of Science, Technology, Education, and Management*, Volume 3, No. 2, June 2023.

Here's the setup: your Design Ruler has three toothed gears and two toothed apertures. The gears have 36, 52, and 63 teeth respectively; the apertures have 96 and 105 teeth respectively.

1. Let's focus on a particular aperture-and-gear combination: the case where the aperture has 96 teeth and the gear has 36 teeth.

   Draw a complete flower design with this combination. Use a light pen color, because soon you are going to be drawing on top of this flower. One you're done, **do not move your template** (that is, the piece of plastic with the apertures). Again, you're going to be drawing on top of your original flower, so you'll want to start with your aperture (and pen and gear) in the same place.

   Also, you'll get a better picture (meaning things will be easier to count) if you place your pen in one of the holes near the perimeter of the gear, rather than one near the center. Make a mental note of which pen hole you're using, because you'll want to use the same one again, soon.

   How many petals does your flower have? Call this number p:

   $p =$ _____

2. What is the *spacing*, or number of teeth (of the aperture), between adjacent petals on your flower? Hint: it's just the number of teeth in the aperture divided by the number of petals. Call this spacing s:

   $s =$ _____

3. Now, using a darker color and starting at the *tip* of one of the petals you drew above (and with your pen in the same hole as previously), retrace your design, until you get to the tip of the petal that's 24 teeth from your starting point. That is: if you think of the teeth in the aperture as being numbered 0 through 95, and you think of the tooth at which you start as being tooth number zero, then keep going until your gear is just touching tooth number 24. Hint: you shouldn't actually have to count out 24 teeth; using your answer for the spacing s between petals, you should be able to figure out which petal tip is 24 teeth beyond the one at which you started.

   As you do this, keep track of two things:

   (a) How many petal tips did you hit in getting to the one above (the one that's 24 teeth from from your starting point)? Answer just by *counting* how many tips you hit, including the last one (the one at which you ended) but not including the zeroth one (the one at which you started). Call this number x, and also compute $36x$:

   $x =$ _____.

   $36x =$ _____.

   (b) How many *complete revolutions* around the inside of the aperture did you make? (You'll have to go around a whole number of times plus a fraction of a revolution to get to the tip in question. Just count the whole number of times.) Call this number y, and also compute $96y$:

   $y =$ _____

   $96y =$ _____

4. Compute $36x - 96y$:

   $36x - 96y =$ _____

5. Fill in the blanks: we have just solve the linear congruence

$$36x - 96y = 24.$$

   Specifically, a solution is given by:

   $x =$ _____, $y =$ _____.

6. Use the above process, and your 63-tooth gear and 105-tooth aperture, to solve the linear congruence

$$63x - 105y = 42 :$$

   $x =$ _____, $y =$ _____.