Expected value, continued.

Recall: for an rv X, we define the expected value $E(X)$ by

$$E(X) = \sum_{\substack{\text{all possible} \\ \text{values } x}} x\, P(X=x).$$

Example 1.

A fair coin is flipped until it lands tails. Let X be the rv that records how many flips it took.

(a) Find the "probability mass function." This just means: find all possible values of $P(X=i)$.

(b) Find $E(X)$.

Solution.

To say that it takes i flips for the coin to land tails is to say that:

(1) The first $i-1$ flips land heads;

(2) The i^{th} lands tails.

Each outcome of a single flip has prob. $1/2$, so

$$P(X=i) = (1/2)^i.$$

So

$$E(X) = \sum_{i=1}^{\infty} i\, P(X=i)$$

$$= \sum_{i=1}^{\infty} i \cdot (1/2)^i \underset{\uparrow}{=} 2.$$

use Wolfram Alpha, Mathematica, Calculus, etc

More on expected value.

Rule 1 (sum rule): for any rv's X and Y,

$$\boxed{E(X+Y)=E(X)+E(Y).}\qquad \text{(Sum Rule)}$$

In general, the expected value of a sum (of any # of rv's) equals the sum of the expected values (of those rv's).

Example 2.

(a) A fair die is tossed. Let X be the number that comes up. Find $E(X)$.

(b) 28 fair dice are tossed. Let Z be the sum of the numbers showing. Find $E(Z)$.

Solution.

(a)
$$\begin{aligned} E(X) &= 1\cdot P(X=1)+2\cdot P(X=2)+\dots+6\cdot P(X=6) \\ &= 1\cdot 1/6 + 2\cdot 1/6+\dots+6\cdot 1/6 = \tfrac{1}{6}(1+2+3+4+5+6) \\ &= 3.5. \end{aligned}$$

(b) Let Z_i be the number showing on the i^{th} die. Then
$$Z = Z_1+Z_2+\dots+Z_{26},$$
so
$$\begin{aligned} E(Z) &= E(Z_1)+E(Z_2)+\dots+E(Z_{26}) \\ &= 26\cdot 3.5 = 91. \end{aligned}$$

Example 3.

In a randomly shuffled deck of cards, how many adjacent pairs of cards would we expect to have the same suit?

Solution.

Call the random variable in question X. If we define rv's $X_1, X_2, X_3, \ldots, X_{51}$ by

$$X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ and } (i+1)^{st} \text{ cards share a suit;} \\ 0 & \text{otherwise,} \end{cases}$$

then

$$X = \sum_{i=1}^{51} X_i.$$

Also, note that $E(X_i) = 12/51$ (whatever suit the i^{th} card is, 12 of the remaining 51 cards have the same suit).

So

$$E(X) = \sum_{i=1}^{51} E(X_i) = 51 \cdot 12/51 = 12.$$

Example 4. Here's a card game:

You draw a card at random from a standard, 52-card deck. You win \$8 if you draw a face card or an ace; you lose \$3 if you draw a 2 through an 8; you lose \$6 if you draw a 9 or a 10.

Should you play?

Solution.

Let W be the rv of how much is won ($W<0$ for a loss).

Then, in dollars,

$$E(W) = 8 \cdot P(\text{win } \$8) - 3 \cdot P(\text{lose } \$3) - 6 \cdot P(\text{lose } \$6)$$

$$= 8 \cdot \frac{4}{13} - 3 \cdot \frac{7}{13} - 6 \cdot \frac{2}{13} = \frac{32-21-12}{13} = -1/13.$$

You'd expect to lose, so you shouldn't play.