Conditional probability, concluded.

We saw in class last time that, for any events A and B,

$$\boxed{P(A) = P(B)P(A|B) + P(B^c)P(A|B^c)}$$ (☺)

where B^c is the complement of B (= {all outcomes not in B}).

A generalization (Rule 8.2 in text):

Suppose $B_1, B_2, B_3, \ldots, B_n$ are mutually exclusive events. Let A be any event that can only happen if one of the B_i's happens. Then

$$\boxed{P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + \ldots + P(B_n)P(A|B_n).}$$

(☺)

Example 1.

You toss a fair coin up to three times. As soon as it lands heads, you're asked a question. You win if you answer correctly and lose if you don't. If you don't win after three tosses, you lose. The questions get easier: you have probability $i/4$ of getting the i^{th} one right. What's the probability of winning?

Solution Consider these events.

A: win

B_1: first toss lands heads

B_2: second does

B_3: third does

Certainly A can only happen if B_1 or B_2 or B_3 happens. So by (☺),

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3).$$

Now we note:

(1) $P(B_1) = \frac{1}{2}$ and $P(A|B_1) = \frac{1}{4}$.

(2) $P(B_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, since you can only get to the second toss if the first is tails. Also, $P(A|B_2) = \frac{2}{4} = \frac{1}{2}$.

(3) $P(B_3) = \frac{1}{8}$, since B_3 means the first two tosses are tails and the third is heads. Also, $P(A|B_3) = \frac{3}{4}$.

So

$$P(A) = \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{8}\cdot\frac{3}{4} = \frac{4}{32} + \frac{4}{32} + \frac{3}{32} = \frac{11}{32}$$

$$\approx 13.67\%.$$

Chapter 9: Random Variables.

A random variable (rv) is, essentially, a way of assigning numbers to outcomes.

Specifically: given an experiment with sample space S, a random variable X is a real-valued function on S.

(A random variable is a function, not a variable!)

Example 2.

If the experiment is to roll two dice, let's write the sample space S in the usual way:

$$S = \{11, 12, \ldots, 16, 21, 22, \ldots, 66\}$$

We can define an rv X on S by
$$X(\omega) = \text{sum of two numbers},$$
for any outcome ω. E.g. $X(15)=6$, $X(43)=7$, etc.

Example 3.

If the experiment is to shoot three free throws, let $S = \{HHH, HHM, HMH, HMM, MHH, MHM, MMH, MMM\}$ (H for hit, M for miss). Define an rv Z on S by

$$Z(\omega) = \#\text{ of hits, for any outcome } \omega.$$
E.g. $Z(MHM)=1$.

Now if X is an rv and x is a real number, we write
$$P(X=x)$$
to denote the probability of the event of all outcomes ω where $X(\omega)=x$.

Example 4

In Example 2 above,

$$P(X=6) = P(\{15, 24, 33, 42, 51\}) = 5/36 \approx 13.89\%,$$

$$P(X=11) = P(\{56, 65\}) = 2/36 \approx 5.56\%,$$
etc.

Example 5.

In Example 3 above, suppose we have an 80% free throw shooter, all of whose shots are independent. Then

$P(Z=3) = P(\{HHH\}) = (0.8)^3 = 0.512$
$P(Z=0) = P(\{TTT\}) = (0.2)^3 = 0.008$

Challenge questions:

$P(Z=1) = ?$ (Answer: 0.096)
$P(Z=2) = ?$ (Answer: 0.384)

Super-challenge:

If n free throws are taken, each with probability p of going in, what is

$$P(\text{exactly } r \text{ go in}),$$

for $0 \le r \le n$??

(Answer: $\binom{n}{r} p^r (1-p)^{n-r}$)