Conditional probability (Chapter 8).

GOAL: to understand "the prob. of A given B," written $P(A|B)$, meaning "the prob. of A happening assuming B happened."

Part 1. A formula for $P(A|B)$.

Suppose we have a sample space S of equally likely outcomes.
We know that

$$P(A) = \frac{n(A)}{n(S)}. \qquad (*_0)$$

To get $P(A|B)$ from this, we have to:

(a) Replace $n(S)$ by $n(B)$, since assuming B happened means only looking at outcomes in B. (That is, B becomes our new sample space.)

(b) Replace $n(A)$ by $n(AB)$ since, if we're assuming B happened, then whenever A happens, both A and B do.

SO: $(*_0)$ gives

$$P(A|B) = \frac{n(AB)}{n(B)}. \qquad (*_1).$$

Example 1.
Two fair dice are rolled. Define events A and B:

A: both dice lands on 6.
B: at least one die lands on 6.

Find $P(A|B)$.

Solution

If we write the outcomes as 11, 12, 13, ..., 66, then they are all equally likely.

Note that

$$B = \{61, 62, 63, 64, 65, 66, 56, 46, 36, 26, 16\}$$

and

$$AB = \{66\}.$$

So by $(*_1)$

$$P(A|B) = \frac{n(AB)}{n(B)} = \frac{1}{11} \approx 0.0909$$
$$= 9.09\%.$$

<u>Note:</u> experimentally, AB happened about 12.5% of the time B happened, not 9.09%. Why was our experimental proportion so <u>high</u>?

Possible answer: <u>bias</u>. If we were <u>eager</u> to <u>stop</u> as soon as we got our 5^{th} (or 6^{th}, or 7^{th}, or...) 66, then we <u>favored</u> that outcome.

Let's go back to $(*_1)$. We'd like a formula <u>only</u> involving probabilities, not sizes of sets. To do this, multiply top and bottom of the right side of $(*_1)$ by $n(S)$:

$$P(A|B) = \frac{n(AB)/n(S)}{n(B)/n(S)} = \frac{P(AB)}{P(B)}.$$

This formula:

$$\boxed{P(A|B) = \frac{P(AB)}{P(B)}} \qquad (*_2)$$

is the formula for the conditional prob. $P(A|B)$.
We define $P(A|B)$ this way even when outcomes aren't equally likely.

Example 2.
A fair coin, and an unfair coin, with $P(\text{heads}) = 1/3$, are tossed. Given that at least one coin lands heads, what is $P(\text{at least one coin lands tails})$?

Solution. Our events are
A: at least one coin lands tails
B: at least one coin lands heads.

By $(*_2)$, $P(A|B) = P(AB)/P(B)$.
Now

$$P(AB) = P(\text{fair coin lands heads and unfair lands tails}) + P(\text{fair coin lands tails and unfair lands heads})$$
$$= \tfrac{1}{2}\cdot\tfrac{2}{3} + \tfrac{1}{2}\cdot\tfrac{1}{3} = \tfrac{2}{6} + \tfrac{1}{6} = \tfrac{3}{6} = \tfrac{1}{2},$$

while

$$P(B) = P(\text{fair is heads, unfair lands tails}) + P(\text{fair is tails, unfair is heads})$$

+ P(both are heads)

$$= \frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3} = 4/6 = 2/3.$$

So

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{1/2}{2/3} = \frac{1}{2}\cdot\frac{3}{2} = 3/4 = 75\%.$$

Question: what would $P(A|B)$ be if both coins are fair??