More examples.

Example 1.
Two socks are selected at random from a drawer containing four red and five green socks. Let A be the event where one red and one green sock are drawn. What is $P(A)$?

Solution A: keep track of order.
Then $n(S) = 9 \cdot 8 = 72$ (9 ways to pick first sock, then 8 remaining ways to pick second). Now note that $A = A_1 \cup A_2$, where A_1 is where the first sock is red and the second green, while A_2 is where the first is green and the second red.

By Axiom 3, $P(A) = P(A_1 \cup A_2) = P(A_1) + P(A_2)$. Also, note that $n(A_1) = 4 \cdot 5 = 20$ (4 ways to pick first sock, then 5 ways to pick second). So

$$P(A_1) = n(A_1)/n(S) = 20/72.$$

Similarly, $n(A_2) = 5 \cdot 4 = 20$, so

$$P(A_2) = n(A_2)/n(S) = 20/72. \text{ So}$$

$$P(A) = P(A_1) + P(A_2) = 20/72 + 20/72 \approx 0.5556 = 55.56\%.$$

Solution B: Don't keep track of order.
Then

$$n(S) = \binom{9}{2} = \frac{9!}{2!7!} = 36.$$

If we (again) define A as the event where the two socks drawn have different colors, then

$$n(A) = \binom{4}{1}\binom{5}{1} = 5 \cdot 4 = 20$$

(4 ways of choosing a red sock, and 5 ways of choosing a green). So by Axiom 4,

$$P(A) = n(A)/n(S) = 20/36 = 0.\overline{5}$$
$$\approx 0.5556 = 55.56\%$$

(same answer as in Solution A).

Example 2.

A generalization of Rule 3 (which says that $P(A \cup B) = P(A) + P(B) - P(AB)$ for any events A, B) says:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$$

(for any events A, B, C).

Suppose a fair die is rolled, and a card is drawn at random from a standard deck of 52 cards. Find P(the die shows a multiple of 3 or the card is either a 10 or a heart).

Solution Call the events in question A, B, C respectively. Then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$$

because A, B, C are independent

$$= P(A) + P(B) + P(C) - P(A)P(B) - P(B)P(C) - P(A)P(C) + P(ABC)$$

$$= \tfrac{1}{3} + \tfrac{4}{52} + \tfrac{13}{52} - \tfrac{1}{3} \cdot \tfrac{4}{52} - \tfrac{4}{52} \cdot \tfrac{13}{52} - \tfrac{1}{3} \cdot \tfrac{13}{52} + \tfrac{1}{3} \cdot \tfrac{4}{52} \cdot \tfrac{13}{52}$$

$\approx 0.5385 = 53.85\%$.

Example 3.

Five cards are drawn at random from a standard deck of 52 cards. Find P(at least one ace is drawn).

Solution

"at least one" means 1, 2, 3, 4, or 5, and considering each of these separately is too much work!

Instead, note that

$$P(\text{at least one ace}) = 1 - P(\text{no aces}).$$

What is P(no aces)? Well, let's count. There are $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ ways of drawing 5 cards (keeping track of order). Of these ways,

$$48 \cdot 47 \cdot 46 \cdot 45 \cdot 44$$

have no aces (keeping track of order). All these ways are equally likely, so

$$P(\text{no aces}) = \frac{48 \cdot 47 \cdot 46 \cdot 45 \cdot 44}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \approx 0.6588.$$

So

$$P(\text{at least one ace}) \approx 1 - 0.6588$$
$$= 0.3412 = 34.12\%.$$

(DIY: do this without keeping track of order.)