Monday, 9/11

Summary of probability axioms, plus some probability rules, plus an example.

A. Notation. Given an experiment, write S for the sample space, or set of possible outcomes (things that can happen).
Use upper-case letters for events (subsets of S).
$P(A)$ denotes the probability of the event A.
$n(A)$ denotes the number of elements (outcomes) of A
(assuming this number is finite).

B) Axioms (for events; similar things happen for outcomes):

1) $P(A) \geq 0$ for any event A (nothing has a negative chance of happening).

2) $P(S) = 1$ (that is, something must happen).

3) meaning: A_1 or A_2 or A_3 or ... happens

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

as long as $A_1, A_2, A_3, \ldots$ are mutually exclusive (no two of these events can happen together).
(Here, $A_1, A_2, A_3 \ldots$ could be a finite or infinite list.)

4) If all outcomes in S are equally likely, then $P(A) = n(A)/n(S)$.

5) If events $A_1, A_2, A_3, \ldots$ (finite or infinite list) are independent (they don't affect each other), then

$$p(A_1A_2A_3\cdots) = p(A_1)\cdot p(A_2)\cdot p(A_3)\cdot\ldots$$

meaning: $A_1, A_2, A_3, \ldots$ all happen. Also written $A_1 \cap A_2 \cap A_3 \cap \ldots$

c) Rules (things that follow from axioms):

1. Same as Axiom 3 above.
2. For any event A,
$$P(A) = 1 - P(A^c),$$
where $A^c = \{$everything in S except what's in $A\}$.
(Other notation for A^c: $S \setminus A$ or $S - A$.)
3. In general,
$$P(A \cup B) = P(A) + P(B) - P(AB).$$
4. Rule 3 extends to more than two sets (examples later).

Example. Three fair coins are tossed. Write $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. Let A be the event where at least two coins land heads, B the event where <2 land heads; C the event where all land heads; and D the event where either the first or second coin lands heads (or both do).

(i) $P(C) = P(\{HHH\}) = 1/8$ by Axiom 4. OR:
$P(C) = P(\text{first coin is heads}) \cdot P(\text{second is heads}) \cdot P(\text{third is heads}) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = 1/8$ by axioms 4 and 5.

(ii) $P(A) = P(\{HHH, HHT, HTH, THH\}) = 4/8$ by Axiom 4. OR:

$$P(A) = P(2 \text{ heads}) + P(3 \text{ heads})$$
$$= P(\{HHT, HTH, THH\}) + P(\{HHH\})$$
$$= 3/8 + 1/8 = 4/8 = 1/2 \text{ by Axioms 3 and 4.}$$

(iii) $P(B) = 1 - P(A) = 1 - 1/2 = 1/2$ by Rule 2.

(iv) $P(D) = P(\text{first coin is heads}) + P(\text{second is heads})$
$\quad - P(\text{both are})$
$\quad = 1/2 + 1/2 - P(\{HHT, HHH\})$
$\quad = 1/2 + 1/2 - 2/8 = 3/4,$

by Axiom 4 and Rule 3.