Probability basics (Sections 7.1 and 7.2): mathematically, it's all about sets (of things that might happen).

A probability model is a mathematical representation of an experiment (meaning: any phenomenon that can happen in various ways).
A prob. model consist of:
(i) A description of all possible outcomes (things that can happen). The set of all possible outcomes is called the sample space, often denoted S (or Ω).

(ii) A way of assigning a number (a probability) to each subset of S.

A subset A of S is called an event. So: an event is a collection of possible outcomes.

The probability of an event A is denoted $P(A)$. Sometimes, we also write $p(\omega)$ for the probability of an outcome ω.
The function P is called a probability measure.

Example 1.
Three fair coins are tossed. The sample space is
$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

If A is the event where at least two coins land heads, then
$$A = \{HHH, HHT, HTH, THH\}.$$

If the coins are <u>fair</u>, then all outcomes are equally likely, so

$$p(\omega) = 1/8$$

for <u>any</u> outcome ω. Also, for A as above,

$$P(A) = 4/8 = 1/2,$$

since A contains 4 of 8 equally likely outcomes.

<u>Example 2.</u>

A dart is thrown at a dartboard. Assume the dart always hits the board, and all parts of the board have the same chance of being hit.

Then

$$S = \{\text{all points on the board}\}.$$

This set is "uncountable:" it has two many elements to count. For any <u>given</u> point ω on the board, we have

$$p(\omega) = 0.$$

But we can still define the prob. of certain events. For example: if R is any <u>region</u> of the board of area A, then

$$p(R) = \frac{\text{area } A}{\text{total area of board.}}$$

Some basic probability assumptions (axioms) in this course:

<u>Axiom 1:</u> $P(A) \geq 0$ for every event A.

<u>Axiom 2:</u> $P(S) = 1$ (where S is the

sample space).

[meaning: A_1 or A_2 or A_3 or ... happens]

Axiom 3:

$$P(\boxed{A_1 \cup A_2 \cup A_3 \cup \ldots}) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

as long as the sets $A_1, A_2, \ldots$ are pairwise disjoint (no two of these events can happen at the same time).

Here, the list $A_1, A_2, \ldots$ may be finite or infinite.

Axiom 3': Suppose $A = \{\omega_1, \omega_2, \omega_3, \ldots\}$ (a finite or infinite list). Then

$$P(A) = p(\omega_1) + p(\omega_2) + p(\omega_3) + \ldots$$

Axiom 4 (called the classic probability model):

Given a finite set A, write $n(A)$ for the number of elements of A. Let S denote the sample space S; suppose S is finite. If all elements of S are equally likely, then

(a) $p(\omega) = 1/n(S)$ for any outcome ω;

(b) $p(A) = n(A)/n(S)$ for any event A.

Example 3. We used Axiom 4(a) implicitly in Example 1 above, when we said $p(\omega) = 1/8$ for any outcome ω.

Also, if $A = \{\text{at least one head}\}$, then we can compute A using either of these methods:

(i) Axioms 3′ and 4a, like this:

$$P(A) = P(\{HHH, HTH, HHT, THH\})$$
$$= \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{4}{8} = \tfrac{1}{2};$$

(ii) Axioms 3 and 4b, like this:

$$\begin{aligned} P(A) &= P(\text{at least two heads}) \\ & P(\text{two heads}) + P(\text{three heads}) \\ &= P(\{HTH, HHT, THH\}) + P(\{HHH\}) \\ &= \tfrac{3}{8} + \tfrac{1}{8} = \tfrac{4}{8} = \tfrac{1}{2}. \end{aligned}$$

<u>Axiom 5:</u>

Suppose we have a <u>compound</u> experiment, meaning: each outcome ω amounts to n sub-outcomes $\omega_1, \omega_2, \ldots, \omega_n$ occuring together. Then, <u>assuming</u> these n outcomes are independent (they don't affect each other), we have:

$$p(\omega) = p(\omega_1) \cdot p(\omega_2) \cdot p(\omega_3) \cdots p(\omega_n)$$

(in fact, this works for infinitely many ω_i's as well).

<u>Example 4:</u> In Example 1 above, any outcome ω is the result of the 3 simple, independent outcomes $\omega_1, \omega_2,$ and ω_3, where ω_i is how the i^{th} coin lands ($i = 1, 2,$ or 3). So for example,

$$p(HTH)$$

$= p(\text{first coin lands heads}) \cdot p(\text{second lands tails})$
$\cdot\, p(\text{third lands heads})$

$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

the coin is fair; use Axiom 4(a)

(and similarly for any outcome).