Informal intro to probability:
some problems, and some tools for studying them.

Part I: some problems (from Chap. 1, with some modifications)

Question 1: What's the probability that, of the 23 people in this class, at least two share a birthday? (Assume there are no Feb. 29 birthdays.)

Poll: do you think this probability is:
(A) $< 50\%$?
(B) $\geq 50\%$?

Votes: (A) 11
(B) 7

Question 5: In a Powerball lottery, where you must match 6 out of 45 different numbers exactly, is your probability of winning

(A) greater than; or
(B) less than

Votes: (A) 9
(B) 10

the probability of a fair coin, flipped 22 times, landing heads each time?

Question 8:
You wash 10 pairs of socks, each pair a different color, and after washing, 6 socks are lost. Which is more likely:

(A) The best case scenario
(seven matching pairs left); or

(B) The worst case scenario
(four matching pairs left).

Votes: (A) 5
(B) 15

Part II: Some solution tools: permutations and combinations. (From the Appendix.)

(i) permutations.

Question: how many ways can n different objects be arranged (in a row)?

E.g. there are six different arrangements - also called permutations - of the letters X, Y, Z:

XYZ, XZY, YXZ, YZX, ZXY, ZYX.

Similarly, there are 24 permutations of the letters W, X, Y, Z:

WXYZ, WXZY, WYXZ, WYZX, WZXY, WZYX, XWYZ XWZY, ...

(DIY, or Do It Yourself: list them all!)

In general, we argue like this: to arrange n objects, we:

pick the first object: n ways
pick the second object: $n-1$ ways
pick the third object: $n-2$ ways

$\vdots$

pick the next-to-last object: 2 ways
pick the last object: 1 way

These probabilities multiply, so all told there are

$$\underbrace{n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1}_{\text{denote this number by } n! \text{ ("} n \text{ factorial")}}$$

ways of arranging n objects.
Convention: we define $0!$ to be 1.

Example: suppose Dr. S. has a different pair of sneakers for each of the 44 class days this semester. If he wears a different pair each day, there are

$$44! = 44 \cdot 43 \cdot 42 \cdots 3 \cdot 2 \cdot 1 \approx 2.66 \times 10^{54}$$

different ways he can do this.
(Note: the universe weighs about 2×10^{52} kg.)

(ii) Combinations.

Question: how many ways can we choose 3 distinct letters out of the 8 letters A, B, C, D, E, F, G, H, where order doesn't matter? (That is: we only care which letters we get, not which got chosen when.)

To answer, reason as follows:

(a) There are 8 ways of choosing the first letter, then 7 ways of choosing the second, then 6 ways of choosing the last, ~~for~~ $8\cdot7\cdot6$ ways of choosing. BUT

(b) We've overcounted, e.g. choosing A, then C, then F is the same as choosing F, then A, then C. To compensate, divide by the number of permutations of A, C, E (or of any other three letters), which is $3!$, by part (i)

CONCLUSION:

There are $\dfrac{8\cdot7\cdot6}{3!}$ ways of choosing 3 distinct letters out of 8.

Remarks:

(a) Note that

$$\frac{8\cdot7\cdot6}{3!} = \frac{8\cdot7\cdot6}{3!}\cdot\frac{5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1} = \frac{8!}{3!\,5!}.$$

(b) More generally, the number of ways of choosing r objects out of n distinct objects is

$$\frac{n!}{r!(n-r)!},$$ denoted $\binom{n}{r}$ ("n choose r.")

(c) $\binom{n}{r}$ is called a binomial coefficient.

(d) For example, Dr. S. can choose 3 pair of sneakers from his 44 pair in

$$\binom{44}{3} = 13{,}244 \text{ ways.}$$