Let's do some HW problems.

Problem 9.3. A bag contains three coins: two fair normal coins, and one two-headed coin. A coin is picked at random from the bag. Find the probability mass function for X, the number of heads that come up.

Solution.

X can equal 0, 1, or 2.

We have

$P(X=0) = P(\text{no heads})$
$= P(\text{no heads and normal coin})$
$+ P(\text{no heads and two-headed coin})$
$= P(\text{normal coin}) \cdot P(\text{no heads given normal})$
$+ P(\text{two-headed}) \cdot P(\text{no heads given two-headed})$
$= 2/3 \cdot 1/4 + 1/3 \cdot 0 = 1/6 \approx 16.67\%.$

$P(X=1) = P(\text{exactly one head})$
$= P(\text{normal coin}) \cdot P(\text{one head given normal})$
$+ P(\text{two-headed}) \cdot P(\text{one head given two-headed})$
$= 2/3 \cdot 1/2 + 1/3 \cdot 0 = 1/3 \approx 33.33\%.$

(If the coin is normal, then two out of four, i.e. 1/2, of the possible outcomes have exactly one head.)

$P(X=2) = P(\text{two heads})$
$= P(\text{normal coin}) \cdot P(\text{two heads given normal})$
$+ P(\text{two-headed}) \cdot P(\text{two heads given two-headed})$
$= 2/3 \cdot 1/4 + 1/3 \cdot 1 = 1/2 = 50\%.$

//

New observation:

Suppose an rv $X$ can be written as a sum $X_1 + X_2 + X_3 + \ldots + X_n$. Suppose each $X_i$ can only equal $0$ or $1$. (For example, $X_i$ might be 1 when a certain thing happens, 0 if not.) Then note the following:

(a) For any $i$,

$$E(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1)$$
$$= P(X_i = 1).$$

(b)
$$E(X) = E(X_1 + X_2 + \ldots + X_n)$$
$$= E(X_1) + E(X_2) + \ldots + E(X_n)$$
$$= P(X_1 = 1) + P(X_2 = 1) + \ldots + P(X_n = 1).$$

In sum: if $X$ is a sum of rv's $X_i$ that equal $0$ or $1$, then $E(X)$ is the sum of the probabilities $P(X_i = 1)$.

In practice, we can use this idea to count the expected number of times $X$ that a certain thing happens.

E.g. we might count:

(a) The expected number of free throws made in 3 attempts;

(b) The expected number of times a basketball player makes 2 consecutive shots in a game.

Or:

Problem 9.17

What is the expected number of times

that two consecutive numbers will show up, if 6 distinct numbers are chosen at random from the numbers $1, 2, 3, \ldots, 45$?

<u>Solution</u>

Call the number of times that two consecutive numbers show up $X$.

Also define

$$X_1 = \begin{cases} 1 & \text{if } 1 \text{ and } 2 \text{ show up;} \\ 0 & \text{if not;} \end{cases}$$

$$X_2 = \begin{cases} 1 & \text{if } 2 \text{ and } 3 \text{ show up;} \\ 0 & \text{if not;} \end{cases}$$

$$\vdots$$

$$X_{44} = \begin{cases} 1 & \text{if } 44 \text{ and } 45 \text{ show up;} \\ 0 & \text{if not.} \end{cases}$$

Note that $X = X_1 + X_2 + \ldots + X_{44}$.

Now what is $P(X_1 = 1)$? This is $P(1 \text{ and } 2 \text{ show up})$. There are $\binom{45}{6}$ ways of choosing 6 numbers from 45. Of these ways, $1 \cdot 1 \cdot \binom{43}{4}$ ways have a 1 and a 2 (1 way of choosing a 1, then 1 way of choosing a 2, then $\binom{43}{4}$ ways of choosing the remaining 4 numbers). So

$$P(X_1 = 1) = \frac{\binom{43}{4}}{\binom{45}{6}}.$$

Similarly, $P(X_i)$ is the same number for $i = 1, 2, \ldots, 44$. So

$$E(X) = E(X_1) + E(X_2) + \ldots + E(X_{44})$$

$$= 44 \cdot \frac{\binom{43}{4}}{\binom{45}{6}} = \frac{2}{3} = 0.6667.$$

(Note: HW problem 9.14 is similar.)