

Monday, 10/6 - (1)

Let's do some HW problems.

Problem 4.3. A bag contains three coins: two fair normal coins, and one two-headed coin. A coin is picked at random from the bag. Find the probability mass function for X , the number of heads that come up.

Solution.

X can equal 0, 1, or 2.

We have

$$\begin{aligned} P(X=0) &= P(\text{no heads}) \\ &= P(\text{no heads and normal coin}) \\ &\quad + P(\text{no heads and two-headed coin}) \\ &= P(\text{normal coin}) \cdot P(\text{no heads given normal}) \\ &\quad + P(\text{two-headed}) \cdot P(\text{no heads given two-headed}) \\ &= \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot 0 = \frac{1}{6} \approx 16.67\%. \end{aligned}$$

$$\begin{aligned} P(X=1) &= P(\text{exactly one head}) \\ &= P(\text{normal coin}) \cdot P(\text{one head given normal}) \\ &\quad + P(\text{two-headed}) \cdot P(\text{one head given two-headed}) \\ &= \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0 = \frac{1}{3} \approx 33.33\%. \end{aligned}$$

(If the coin is normal, then two out of four, i.e. $\frac{1}{2}$, of the possible outcomes have exactly one head.)

$$\begin{aligned} P(X=2) &= P(\text{two heads}) \\ &= P(\text{normal coin}) \cdot P(\text{two heads given normal}) \\ &\quad + P(\text{two-headed}) \cdot P(\text{two heads given two-headed}) \\ &= \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot 1 = \frac{1}{2} = 50\%. \end{aligned}$$

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New observation:

Suppose an rv X can be written as a sum $X_1 + X_2 + X_3 + \dots + X_n$. Suppose each X_i can only equal 0 or 1. (For example, X_i might be 1 when a certain thing happens, 0 if not.) Then note the following:

$$(a) \text{ For any } i, \\ E(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) \\ = P(X_i = 1).$$

$$(b) \begin{aligned} E(X) &= E(X_1 + X_2 + \dots + X_n) \\ &= E(X_1) + E(X_2) + \dots + E(X_n) \\ &= P(X_1 = 1) + P(X_2 = 1) + \dots + P(X_n = 1). \end{aligned}$$

In sum: if X is a sum of rv's X_i that equal 0 or 1, then $E(X)$ is the sum of the probabilities $P(X_i = 1)$.

In practice, we can use this idea to count the expected number of times X that a certain thing happens.

E.g. we might count:

- (a) The expected number of free throws made in 3 attempts;
- (b) The expected number of times a basketball player makes 2 consecutive shots in a game.

Or:

Problem 9.17

What is the expected number of times

that two consecutive numbers will show up, if 6 distinct numbers are chosen at random from the numbers $1, 2, 3, \dots, 45$?

Solution

Call the number of times that two consecutive numbers show up X .

Also define

$$X_1 = \begin{cases} 1 & \text{if 1 and 2 show up;} \\ 0 & \text{if not;} \end{cases}$$

$$X_2 = \begin{cases} 1 & \text{if 2 and 3 show up;} \\ 0 & \text{if not;} \end{cases}$$

$\vdots$

$$X_{44} = \begin{cases} 1 & \text{if 44 and 45 show up;} \\ 0 & \text{if not.} \end{cases}$$

Note that $X = X_1 + X_2 + \dots + X_{44}$.

Now what is $P(X_1 = 1)$? This is $P(1 \text{ and } 2 \text{ show up})$. There are $\binom{45}{6}$ ways of choosing 6 numbers from 45. Of these ways, $1 \cdot 1 \cdot \binom{43}{4}$ ways have a 1 and a 2 (1 way of choosing a 1, then 1 way of choosing a 2, then $\binom{43}{4}$ ways of choosing the remaining 4 numbers). So

$$P(X_1 = 1) = \frac{\binom{43}{4}}{\binom{45}{6}}.$$

Similarly, $P(X_i)$ is the same number for $i = 1, 2, \dots, 44$. So

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$$E(X) = E(X_1) + E(X_2) + \dots + E(X_{44})$$

$$= 44 \cdot \frac{\binom{43}{4}}{\binom{45}{6}} = \frac{2}{3} = 0.6667.$$

(Note: tlw problem 9.14 is similar.)