Homework Assignment #9: The Poisson Distribution

Please complete and turn in on Wednesday, December 13. You may write your answers on these pages or on your own paper.

Recall the following. Suppose a certain event happens, on average, λ times per interval of a given extent. Let X be the number of times that the event *actually* happens in such an interval. Then we say "X is Poisson of parameter λ," and we have the formula

$$P(X = k) = \frac{\lambda^k}{k!}\, e^{-\lambda} \qquad (k = 0, 1, 2, 3, \ldots). \tag{$*$}$$

For the first four problems we refer to a 1911 experiment of Ernest Rutherford et al., who determined that, on average, a certain sample of polonium emitted 3.8715 α-rays every 7.5 seconds.

1. Find the probability that, in a 7.5-second interval, this sample emits 6 α-rays.

2. Fill in the blanks: Note that $4 \times 7.5 =$ ____________. So, since an average of 3.8715 α-rays are emitted every 7.5 seconds, we would expect that, on average, $4 \times 3.8715 =$ _______________ α-rays will be emitted every 30 seconds.

3. Let Y be the number of α-rays emitted in a 30 second interval. Using your answer to question 2 above, find $P(Y = 24)$. Hint: your parameter λ here is four times as large as in question 1 above.

4. True or false: the probability of getting 24 α-rays in 30 seconds is the same as the probability of getting 6 α-rays in 7.5 seconds. Explain how you know. Is this result surprising to you? If so, why? If not, why?

5. In class on Wednesday we saw that, for a Poisson distribution of parameter $\lambda = 10$, two of the bars had the same height. In other words we saw that, for a certain integer k, and for $\lambda = 10$, $P(X = k) = P(X = k + 1)$.

   Find k. Hint: use formula $(*)$, from the first page of this worksheet, with $\lambda = 10$, to write down formulas for $P(X = k)$ and $P(X = k + 1)$. Set them equal and solve for k.

In discussing the Poisson distribution in class on Monday, we first came up with the formula

$$P(X = k) = \lim_{N\to\infty} \binom{N}{k} \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k}, \qquad \textbf{(**)}$$

and then showed that, at least for $k = 0, 1$, or 2, **(**)** yields **(*)**. The purpose of the next few exercises is to show that this holds for $k = 3$ as well.

6. Show that

$$\binom{N}{3} = \frac{N(N-1)(N-2)}{3!}.$$

Hint: start with the definition $\binom{N}{3} = \frac{N!}{3!\,(N-3)!}$. Now simplify the quantity $\frac{N!}{(N-3)!}$.

7. Use your answer to question 6 above, together with (∗∗), to show that

$$P(X = 3) = \frac{\lambda^3}{3!} \lim_{N \to \infty} \frac{N(N-1)(N-2)}{N^3} \left(1 - \frac{\lambda}{N}\right)^{N-3}.$$

8. Use your answer to question 7 above to find a simple formula (with no limits in it) for $P(X = 3)$. Hint: use the following limit formulas (which may be proved using calculus ideas):

$$\lim_{N\to\infty} \frac{N(N-1)(N-2)}{N^3} = 1; \quad \lim_{N\to\infty} \left(1 - \frac{\lambda}{N}\right)^N = e^{-\lambda}, \quad \lim_{N\to\infty} \left(1 - \frac{\lambda}{N}\right)^{-3} = 1.$$

Another hint: you know what your answer should be, by $(*)$.