Solutions to Selected Exercises, HW #2

Part A: Problems 7.2, 7.3, 7.4, 7.6, 7.7, 7.8.

Problem 7.2. In a township, there are two plumbers. On a particular day three residents call village plumbers independently of each other. Each resident randomly chooses one of the two plumbers. What is the probability that all three residents will choose the same plumber?

SOLUTION. Label the residents R_1, R_2, and R_3, say, and label the plumbers P and O (for "Plumber" and "Other Plumber"). Then the sample space S is

$$S = \{PPP, PPO, POP, POO, OPP, OPO, OOP, OOO\}.$$

For example, POP means resident R_1 chose plumber P, resident R_2 chose plumber O, and resident R_3 chose plumber P. Since the residents are choosing independently, all eight outcomes are equally likely.

The event A, where all three residents chose the same plumber, is the set

$$A = \{PPP, OOO\}.$$

So

$$P(A) = n(A)/n(S) = 2/8 = 1/4.$$

Problem 7.3. Four black socks and five white socks lie mixed up in a drawer. You grab two socks at random from the drawer. What is the probability of having grabbed one black sock and one white sock?

SOLUTION. We did this in class, except with green and red socks instead of black and white. Let's do it again. We won't keep track of the order in which the socks were drawn, though you can do this by keeping track as well. (See Example 1 from our Wednesday, 9/13 lecture notes.)

The number of ways of drawing 2 socks from a drawer of 9 socks is $\binom{9}{2} = 9!/(2!\,7!) = 36$. The number of ways of doing this where one sock is black and the other white is $\binom{4}{1} \cdot \binom{5}{1} = 4 \cdot 5 = 20$. So the probability of drawing one black sock and one white sock is

$$\frac{20}{36} \approx 0.5556 = 55.56\%.$$

Problem 7.4. Two letters have fallen out of the word Cincinnati at random places. What is the probability that these two letters are the same?

SOLUTION. There are 10 letters in Cincinnati, so the number of ways in which 2 letters can fall out is $\binom{10}{2} = 45$. How many ways are there where the 2 letters are the same? Well, let A be the event where both letters are c (we'll assume that lower and upper case c's are the same), let B be the event where both letters are i, and let C be the event where both letters are n. There's only 1 way for A to happen, since there are only 2 c's. There are $\binom{3}{2} = 3$ ways for B to happen, since there are $\binom{3}{2}$ ways to choose 2 i's from among the 3 in Cincinnati. Similarly, there are $\binom{3}{2} = 3$ ways for C to happen. Also, these events are mutually exclusive, since there aren't enough letters falling out for more than one of these things to happen at once.

So

$$\begin{aligned} &P(\text{the two letters that fall out are the same}) \\ &= \frac{n(A \cup B \cup C)}{n(S)} = \frac{n(A) + n(B) + n(C)}{n(S)} = \frac{1+3+3}{45} = \frac{7}{45} \approx 0.155556 = 15.56\%. \end{aligned}$$

In the above solution, we didn't keep track of order. (For example, in saying that the sample space has 45 elements, we were just thinking of choosing 2 out of 10 letters, we weren't thinking "choose one letter, then choose the other.") But we could keep track of order. If we do so, then our sample space S (note that this S is technically different from the one in the solution above) has $10 \cdot 9 = 90$ elements (10 ways for the first letter to fall out, then 9 ways for the second). Also, There are $2 \cdot 1 = 2$ ways of choosing one c and then another; $3 \cdot 2 = 6$ ways of choosing one i and then another; and $3 \cdot 2 = 6$ ways of choosing one n and then another. So

$$\begin{aligned} &P(\text{the two letters that fall out are the same}) \\ &= \frac{2+6+6}{90} = \frac{14}{90} \approx 0.155556 = 15.56\%, \end{aligned}$$

again.

Just for fun, let's solve this problem one more way: we note that

$$\begin{aligned} &P(\text{the two letters that fall out are the same}) \\ &= 1 - P(\text{the two letters that fall out are different}). \end{aligned}$$

We'll keep track of order again, so again, our sample space has size 90. Then how many ways can the two letters be different? Well, the first letter that falls out could be c. There are 2 ways for this to happen, and if it does, the second letter can't be a c, so there are only 8 possible choices for the second letter. Similarly, there are 3 ways for the first letter to be an i, and for each of these ways, there are only 7 choices for the second letter. Next: there are 3 ways for the first letter to be an n, and for each of these ways, there are only 7 choices for the second letter. There is 1 way for the first letter to be an a, and then there are 9 choices for the second letter. Finally, there is 1 way for the first letter to be a t, and then there are 9 choices for the second letter. These are the only possibilities. So

$$\begin{aligned} &P(\text{the two letters that fall out are the same}) \\ &= 1 - P(\text{the two letters that fall out are different}) \\ &= 1 - \frac{2 \cdot 8 + 3 \cdot 7 + 3 \cdot 7 + 1 \cdot 9 + 1 \cdot 9}{90} = 1 - \frac{76}{90} \approx 0.155556 = 15.56\%, \end{aligned}$$

again.

Problem 7.6. Independently of each other, two people think of a number between 1 and 10. What is the probability that five or more numbers will separate the two numbers chosen at random by the two people?

SOLUTION. First of all, it's not entirely clear what "five or more numbers will separate the two numbers" means. It could mean:

(a) There are actually five or more whole numbers between the two chosen numbers. For example, if one number is 2 and the other is 8, then there are five whole numbers separating these two numbers, since the numbers 3, 4, 5, 6, and 7 separate 2 and 8. Or

(b) The distance between the numbers is at least 5. For example, the distance between 8 and 3 is 5.

These are different interpretations. Let's go with (a) first, then we'll look at (b). (You will get credit for either interpretation.)

We'll keep track of order, meaning an (8,2) (first player draws an 8 and second a 2) will be considered different from a (2,8) (first player draws a 2 and second an 8). Then the sample space S has size 100 (10 numbers for the first player to choose, then 10 numbers for the second). And here are the outcomes where five or more numbers separate the two numbers (as in interpretation (a) above):

$$(1,7),(1,8),(1,9),(1,10),(2,8),(2,9),(2,10),(3,9),(3,10),(4,10),$$
$$(7,1),(8,1),(9,1),(10,1),(8,2),(9,2),(10,2),(9,3),(10,3),(10,4).$$

There are 20 such outcomes, so

$$P(\text{at least five numbers separate the two chosen}) = \frac{20}{100} = 20\%.$$

If we use interpretation (b), then there are more outcomes to keep track of. I leave it to you to compute that, in this case,

$$P(\text{at least five numbers separate the two chosen}) = \frac{30}{100} = 30\%.$$

Problem 7.7. You have four mathematics books, three physics books, and two chemistry books. The books are put in random order on a bookshelf. What is the probability of having the books ordered per subject on the bookshelf?

SOLUTION. There are 9 books total. We imagine lining up the books one after the other. Then the sample sample space has size $9 \cdot 8 \cdot 7 \cdot 6 \cdots 2 \cdot 1 = 9!$ There are $4!$ ways of lining up the 4 math books together, $3!$ ways of lining up the 3 physics books together, and 2! ways of lining up the 2 chemistry books together. Now imagine we have put the books into their groups: since there are 3 groups, there are 3! ways of ordering the groups on the shelf (for example, chemistry books then math books then physics books). So there are $4! \cdot 3! \cdot 2! \cdot 3!$ ways of ordering the books per subject (that is, with books of the same subject grouped together). So

$$P(\text{books are ordered per subject}) = \frac{4! \cdot 3! \cdot 2! \cdot 3!}{9!} \approx 0.00476 = 0.476\%.$$

Problem 7.8. Three friends go to the cinema together on a weekly basis. Before buying their tickets, all three friends toss a fair coin into the air at once. If one of the three gets a different outcome than the other two, that one pays for all three tickets; otherwise, everyone pays his own way. What is the probability that one of the three friends will have to pay for ll three tickets?

SOLUTION. The sample space S is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

(hopefully, the meaning is clear). The event A where one of the three friends has a different result from the other two is

$$A = \{HHT, HTH, HTT, THH, THT, TTH, \}.$$

All of the outcomes in S are equally likely, since the coins are fair. So

$$P(\text{one of the friends pays for everything}) = \frac{n(A)}{n(S)} = \frac{6}{8} = 0.75 = 75\%.$$

Part B: Sample space worksheet.

SOLUTION: see next page. (ANSWERS IN RED.)

In this worksheet, we look at several ideas from probability: outcomes; sample space; events; probabilities of outcomes and events.

Consider the experiment where you roll two dice – say, a red die and a white die – and write down a pair of digits between 1 and 6, corresponding to the numbers that come up on the red and the white die respectively. For example, if the red die shows a 5 and the white die shows a 3, then the recorded outcome would be 53.

1. Write down the sample space S for this experiment (that is, the set of all possible outcomes). Don't just describe the sample space in words; instead, list all possible outcomes (for example, 53 is one outcome). Use proper set notation. (This sample space is relatively large. Be patient.)

$$S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36,$$
$$41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}.$$

Here we are keeping track of order; that is, we're thinking of "what comes up on the first die followed by what comes up on the second." If we don't keep track of order, then computations get trickier, because not all outcomes are equally likely. For example, if we only keep track of what two numbers come up, and not the order they come up in, then the outcome where we get a 1 and a 5 is more likely than the outcome where we get a 6 and a 6. (There are two ways to get a 1 and a 5 but only one way to get a 6 and a 6.)

2. Let's assume that the dice are fair, so that the outcomes of your experiment are all equally likely. What **is** the probability, let's call it $p(\omega)$, of any outcome ω? Write it as a fraction. Hint: the two dice are independent of each other.

$p(\omega) =$ $\underline{\quad 1/36 \approx 0.0278 \quad}$.

Explain how you know this. (Refer to results from the text and/or class notes, if you wish.)

All outcomes are equally likely, so the probability of any one outcome is $1/n(S)$, where $n(S)$ is the size of the sample space.

Now let A_2 be the event where the *sum* of the numbers appearing on the dice equals two, A_3 the event where the sum equals three, A_4 the event where the sum equals four, and so on.

3. Write down each of the sets $A_2, A_3, A_4, \ldots, A_{12}$, by listing the elements in each (much as you did for the sample space S above):

$A_2 = \{11\}$

$A_3 = \{12, 21\}$

$A_4 = \{13, 22, 31\}$

$A_5 = \{14, 23, 32, 41\}$

$A_6 = \{15, 24, 33, 42, 51\}$

$A_7 = \{16, 25, 34, 43, 52, 61\}$

$A_8 = \{26, 35, 44, 53, 62\}$

$A_9 = \{36, 45, 54, 63\}$

$A_{10} = \{46, 55, 64\}$

$A_{11} = \{56, 65\}$

$A_{12} = \{66\}$

4. Compute $P(A_2), P(A_3), P(A_4), \ldots$, up to $P(A_{12})$. Hint: since we're in the equally likely case here,

$$\mathrm{P}(A_3) = \frac{\text{number of ways the dice can sum to three}}{\text{size of sample space}},$$

etc. As above, express your answers as fractions (you don't need to reduce them). Hint for checking your work: the probabilities you get must add up to one.

$P(A_2) = 1/36 \approx 0.0278$

$P(A_3) = 2/36 \approx 0.0556$

$P(A_4) = 3/36 \approx 0.0833$

$P(A_5) = 4/36 \approx 0.1111$

$P(A_6) = 5/36 \approx 0.1389$

$P(A_7) = 6/36 \approx 0.1667$

$P(A_8) = 5/36 \approx 0.1389$

$P(A_9) = 4/36 \approx 0.1111$

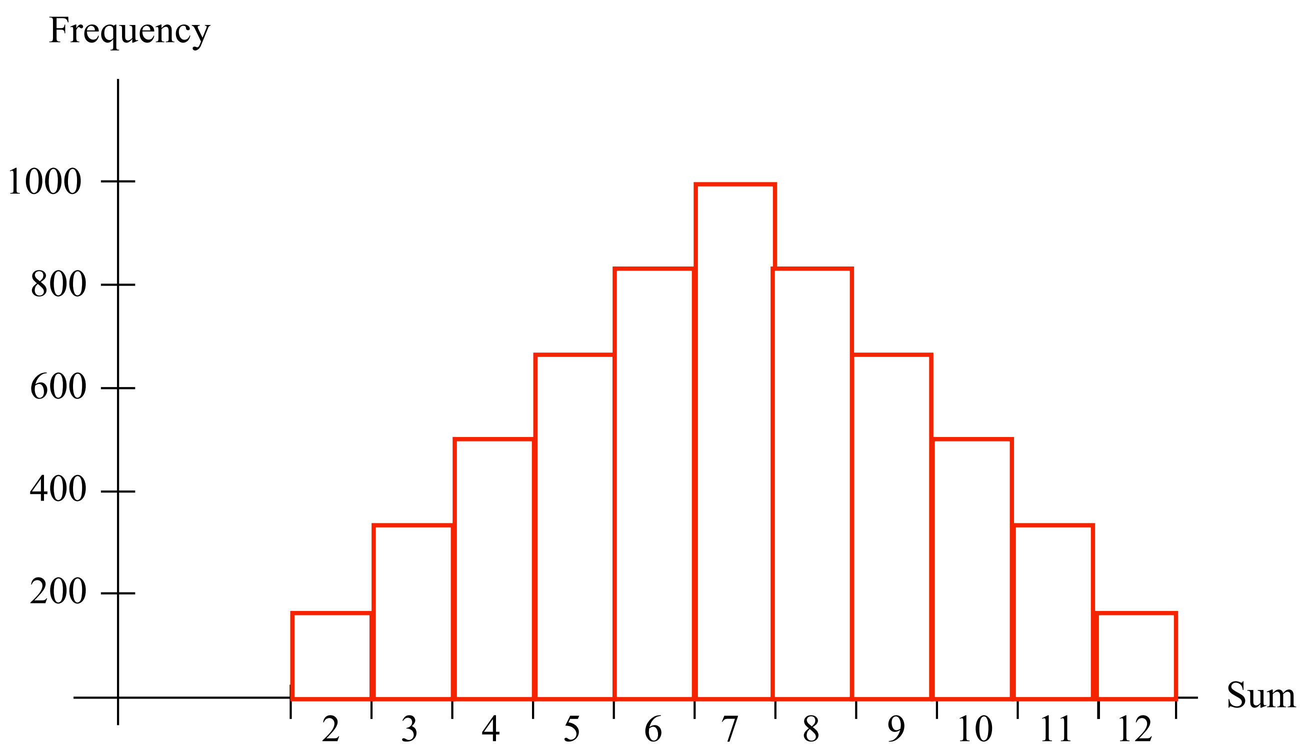
$P(A_{10}) = 3/36 \approx 0.0833$

$P(A_{11}) = 2/36 \approx 0.0556$

$P(A_{12}) = 1/36 \approx 0.0278$

The fractions add up to $(1+2+3+4+5+6+5+4+3+2+1)/36 = 36/36 = 1$, as they should. The decimials actually add up to 1.0001; there's some roundoff error.

5. Suppose you were to repeat the above experiment, of rolling two dice, 6000 times, and each time you record the *sum* of the two numbers that come up. On the axes given below, draw a histogram (like the one we did on class on Friday) of what you think the results would look like, more or less. Please explain your results, in just a sentence or two (or three or four).



Each event would occur approximately in proportion to its probability. For example, we saw above that the event A_6 of getting a sum of 6 has probability 0.1389. So out of 6000 trials, we'd expect to get a sum of 6 about $0.1389 \times 6000 \approx 833$ times. Of course, in "real life" there will be some statistical deviation from the expected results.