Solutions to Selected Exercises, HW #1

Exercise (1) Which is more likely: (a) hitting the jackpot (matching all seven numbers) in a lottery, where seven different numbers are drawn from a set of 48 numbers, or (b), tossing a fair coin 25 times, and getting all tails? Please explain thoroughly, and show all work. NOTE: except for different numbers, this is just like Question 5, page 23, which we discussed in class.

SOLUTION: There are $\binom{48}{7}$ equally likely ways of choosing seven numbers out of 48, and only one of those choices will exactly match the seven winning numbers. So

$$P(\text{winning the lottery}) = \frac{1}{\binom{48}{7}} \approx 1.35816 \cdot 10^{-8}.$$

On the other hand, the probability of a fair coin landing heads on a single toss is 1/2, so the probability of this happening 25 times in a row is

$$P(25 \text{ out of } 25 \text{ tosses land heads}) = \left(\frac{1}{2}\right)^{25} \approx 2.98023 * 10^{-8}.$$

So having the coin land heads 25 out of 25 times is more likely.

Exercise (2) You wash 14 pairs of socks, each pair a different color from the other pairs, and realize, after taking everything out of the dryer, that eight socks are lost. So you are left with $(14 \cdot 2) - 8 = 20$ socks. Find the probability that:

A. You are left with ten matching pairs (this is the best case scenario);

B. You are left with six matching pairs (this is the worst case scenario). Please explain thoroughly, and show all work.

NOTE: except for different numbers, this is just like Question 8, page 23, which we discussed in class.

SOLUTION: There are $\binom{28}{8}$ equally likely ways of choosing eight socks from the 28 total socks. How many of these ways yield the best case scenario? Well, to be left with ten matching pairs is to say that four of the original 14 pairs were lost. There are $\binom{14}{4}$ ways of choosing four pairs from the 14 total pairs. So

$$P(\text{best case scenario}) = \frac{\binom{14}{4}}{\binom{28}{8}} \approx 0.000322061.$$

Now how many of the $\binom{28}{8}$ ways of choosing eight socks yield the worst case scenario? Well, to be left with six matching pairs is to say that eight of the original pairs were unpaired, meaning the eight socks lost consist of one sock of each color. There are $\binom{14}{8}$ ways of choosing the eight colors, and for each color chosen, there are two ways of choosing a sock of that color. So

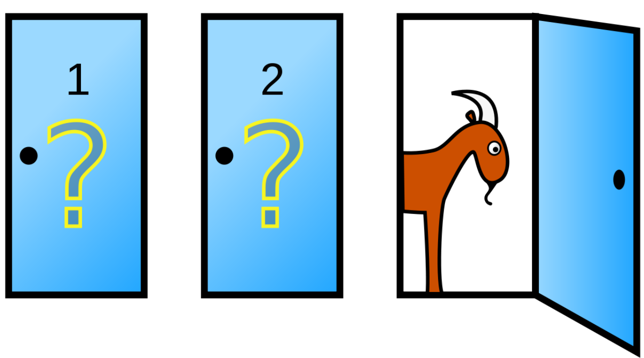
$$P(\text{worst case scenario}) = \frac{\binom{14}{8} \cdot 2^8}{\binom{28}{8}} \approx 0.247343.$$

So $P(\text{worst case scenario}) > P(\text{best case scenario})$ (by a lot).

Exercise (2) (For the the solutions, see the filled-in blanks, in RED, below.) In this problem set, we consider "the Monty Hall Problem," which is a probabilistic curiosity.

Consider this hypothetical scenario from "Let's Make a Deal," a popular TV game show that ran for decades starting in the sixties, hosted by Monty Hall:

> A contestant is shown three doors, and told that there is a new car behind one of them, and a goat behind each of the other two. The contestant chooses a door. Monty Hall **does not** open the chosen door, but **does** reveal a goat behind one of the two doors the contestant did *not* choose.



> Monty Hall then asks the contestant, "Would you like to stick with your first choice, and take whatever lies behind the door you just picked, or instead switch, and go with whatever lies behind the other closed door?"

QUESTION: is the contestant better off switching, or sticking with their original choice? ("Better off" means "more likely to win the car.")

Intuitively, it seems like it shouldn't matter – the contestant has no information about what's behind either of the two closed doors. Right?

Well, let's see. To answer this question, we consider each strategy (STICKING and SWITCHING) separately, and calculate the probability of winning in each case. To this end, let's give names to the three doors: let's call the door with the car behind it W, and call the other two doors L_1 and L_2. Of course, the contestant doesn't know which door is which, though the contestant **does** know, once Monty reveals a goat behind one of the doors, that this door (the one Monty opens) is either door L_1 or door L_2.

Part A: STICKING strategy. This is the strategy where the contestant DOES NOT SWITCH after a goat is revealed behind one of the doors.

Fill in the blanks (there are three of them): Since the contestant is **not** going to switch, the contestant will win if their original pick is the $\underset{(W,\ L_1,\text{ or } L_2)}{\underline{\quad W \quad}}$ door, and will lose otherwise. Since there are **three** doors, and only **one** of them is the $\underset{(W,\ L_1,\text{ or } L_2)}{\underline{\quad W \quad}}$ door, the probability that the contestant will win, with this STICKING strategy, is therefore equal to $\underset{\text{(a fraction, like 1/2, 4/7, etc.)}}{\underline{\quad 1/3 \quad}}$.

Now we consider

Part B: SWITCHING strategy. This is the strategy where the contestant DOES SWITCH after a goat is revealed behind one of the doors. There are three cases to consider here:

Case (i). Fill in the blank (there is just one): Suppose the door the contestant picks in the first place (before seeing the goat) is the W door. Then, since the contestant **is going to switch**, the contestant will $\underset{\text{(win, lose)}}{\underline{\quad \text{lose} \quad}}$.

Case (ii). Fill in the blanks (there are three of them): Now suppose the door that the contestant picks in the first place is the L_1 door. Monty then reveals a goat behind one of the remaining two doors, so that door (the one Monty opens) *must* be the $\underset{(W,\ L_1,\text{ or } L_2)}{\underline{\quad L_2 \quad}}$ door. So the remaining door – the one the contestant is going to switch to – must be the $\underset{(W,\ L_1,\text{ or } L_2)}{\underline{\quad W \quad}}$ door. CONCLUSION: if the first door the contestant picks is the L_1 door, then by switching, the contestant is guaranteeed to $\underset{\text{(win, lose)}}{\underline{\quad \text{win} \quad}}$.

Case (iii). Fill in the blanks (there are three of them): Now suppose the door that the contestant picks in the first place is the L_2 door. Monty then reveals a goat behind one of the remaining two doors, so that door (the one Monty opens) *must* be the $\underline{\quad L_1 \quad}$ (W, L_1, or L_2) door. So the remaining door – the one the contestant is going to switch to – must be the $\underline{\quad W \quad}$ (W, L_1, or L_2) door. CONCLUSION: if the first door the contestant picks is the L_2 door, then by switching, the contestant is guaranteed to ___win___ (win, lose).

SUMMARY of SWITCHING strategy: there are three doors total, and under the SWITCHING strategy, precisely ___2___ (0, 1, 2, or 3) of these doors will yield a win if chosen in the first place. So the probability of winning, under the SWITCHING strategy, is ___2/3___ (a fraction, like 1/2, 4/7, etc.) .

Part C: Comparison of the two strategies. Fill in the blanks (there are three of them): In **Part A** above we saw that, with the STICKING strategy, the contestant's probability of winning is ___1/3___ (a fraction, like 1/2, 4/7, etc.) . On the other hand, in **Part B** above we saw that, with the SWITCHING strategy, the contestant's probability of winning is ___2/3___ (a fraction, like 1/2, 4/7, etc.) . Therefore the ___SWITCHING___ (STICKING, SWITCHING) strategy is the better strategy for winning.

Part D: Reflection. Looking back, can you now explain, intuitively (using as little actual math as possible), why one strategy is better than the other?

We said, above: "Intuitively, it seems like it shouldn't matter – the contestant has no information about what's behind either of the two closed doors." But the contestant *does* have information, indirectly. The contestant knows that Monty Hall knows which door hides the car and which doesn't. And that information, as seen above, makes the switching strategy a better strategy.