GOAL: To use Spirographs to model, and solve, the congruence

$$ax \equiv b \pmod{k}$$

for x (given a, b, and k).

Your Spiral Design Tool (it's NOT a toy, this is SERIOUS!!) has three toothed gears and two toothed apertures. The gears have 36, 52, and 63 teeth respectively; the apertures have 96 and 105 teeth respectively.

1. Compute (a, k) (the greatest common divisor of a and k), where a denotes the number of teeth in a gear and k the number of teeth in an aperture, for all six possible pairings of gear and aperture. Write down your answers in the space below.

2. By experimenting (use your Spiro Tool and some scratch paper), find a gear-and-aperture combination that yields flowers with a relatively small number of petals. (Note: once you've chosen a gear and an aperture, the number of petals you get will NOT depend on which pen-hole you place your pen in.) Let's call the number of petals p.

   (a) Write down a, k, and p (with a and k as described in exercise 1 above) for your choice of gear and aperture.

   (b) Repeat part (a) with a *different* gear-and-aperture combination, and again write down a, k, and p.

3. A fairly simple formula for p in terms of k and (a, k) should now present itself. What *IS* this formula?

4. Using the formula from exercise 3 above, write down the number of petals you'll get for the remaining gear-and-aperture combinations. (That is, the ones you did not address in exercise 2.) (You can check your work by actually drawing the flowers, if you want.)

5. Based on your answer from exercise 3, determine the *spacing*, in terms of number of teeth in the aperture, between adjacent petals. Express your answer in terms of a, k, and (a, k) only. That is: if the tip of a given petal is aligned with a certain tooth, call it tooth number b, on the aperture, and the tip of an adjacent petal is aligned with tooth number $b + \ell$, then what is ℓ? Explain. (Don't just count; explain based on your answer from exercise 3.)

OK, now, HERE'S how we use Spirographs to model linear congruences. The BIG idea is this: we can think of an aperture with k teeth as a model for $\mathbb{Z}/k\mathbb{Z}$; once such a k is chosen, we can think of a gear with a teeth as a model for the equivalence class $\overline{a} = a + k\mathbb{Z}$ in $\mathbb{Z}/k\mathbb{Z}$.

We're going to begin by considering the equivalence class $\overline{36}$ in $\mathbb{Z}/96\mathbb{Z}$. So, be ready with your 36-toothed gear and your 96-toothed aperture.

6. Draw a complete flower design with this gear and aperture. Use a light color, because you're going to be tracing over part of your flower soon. You'll get a better picture if you place your pen in one of the holes near the perimeter of the gear, rather than one near the center.

7. Now, using a darker color and starting at the *tip* of one of the petals you drew above (and with your pen in the same hole as above), retrace your design, until you get to the tip of the petal that's 84 teeth from your starting point. That is: if you think of the teeth in the aperture as being numbered 0 through 95, and you think of the tooth at which you start as being tooth number zero, then keep going until your gear is just touching tooth number 84. Hint: you shouldn't actually have to count to 84; using exercise 5 above, you should be able to figure out which petal tip is 84 teeth beyond the one at which you started.

   Question: How many petal tips did you hit in getting to the one above (the one that's 84 teeth from from your starting point)? Answer just by *counting* how many tips you hit, including the last one (the one at which you ended) but not including the zeroth one (the one at which you started). Call this number x:

   $x =$ _____.

8. Explain *carefully* why your number x from exercise 7 solves the congruence $36x \equiv 84 \pmod{96}$. Hint: drawing from the tip of one petal to the tip of another causes your gear to rotate around its own center exactly once, meaning it corresponds to the meshing of 36 teeth of gear against aperture.

9. Check your work: that is, for the number x you found above, compute $36x - 84$, and show that your result is divisible by 96.

10. Repeat exercises 6–9 for the congruence $52x \equiv 92 \pmod{96}$.

11. Repeat exercises 6–9 for the congruence $36x \equiv 93 \pmod{105}$.

12. Do you think the congruence $36x \equiv 94 \pmod{105}$ has a solution? Explain, in terms of Spirographs.