Multiplicative functions.

Definition.
An arithmetic function f is multiplicative if f is not identically zero and
$$(m,n)=1 \Rightarrow f(mn)=f(m)f(n).$$

Examples:
1) φ is multiplicative by Thm. 2.5(c).
2) The unit function $u \equiv 1$ is multiplicative.

More examples later.

Note that, if f is multiplicative then, choosing $n \in \mathbb{Z}_+$ with $f(n) \neq 0$, we have
$$f(n) = f(n \cdot 1) = f(n) f(1)$$ or, dividing, $f(1)=1$. (Thm 2.12.)

Also:
Thm. 2.14.
f,g multiplicative $\Rightarrow f * g$ is multiplicative.

Proof: next time.

Exercise: recalling that the inverse f^{-1} under $*$ of an $f: \mathbb{Z}_+ \to \mathbb{C}$ with $f(1) \neq 0$ is given by

$$f^{-1}(n) = \begin{cases} 1/f(1) & \text{if } n=1, \\ -\dfrac{1}{f(1)} \displaystyle\sum_{\substack{d \mid n \\ d<n}} f^{-1}(d) f(n/d) & \text{if } n>1, \end{cases}$$

find $f^{-1}(n)$ for $1 \le n \le 8$, $n = 14$, and $n = 28$, if f is the multiplicative function defined on prime powers by

$$f(p^\alpha) = \begin{cases} -1-p & \text{if } \alpha = 1, \\ p & \text{if } \alpha = 2, \\ 0 & \text{if } \alpha > 2. \end{cases}$$

Answers:

$f^{-1}(1) = 1$	$f^{-1}(6) = 12$
$f^{-1}(2) = 3$	$f^{-1}(7) = 8$
$f^{-1}(3) = 4$	$f^{-1}(8) = 15$
$f^{-1}(4) = 7$	$f^{-1}(14) = 24$
$f^{-1}(5) = 6$	$f^{-1}(28) = 56$

Based on the pattern, we conjecture that

$$f^{-1}(n) = \text{sum of the positive divisors of } n = \sum_{d|n} d.$$

Proof of conjecture.

Let $g(n) = \sum_{d|n} d$.

Dirichlet inverses are unique, so if we can show that $f * g = I$, we'll be done.

Now $g = N * u$ where $N(n) = n$ and $u(n) = 1 \ \forall n$. Clearly N and u are multiplicative, so g is by the above theorem, so $f * g$ is by the same theorem.

So it's enough to show that $f*g(p^\alpha) = I(p^\alpha)$ for p prime and $\alpha \geqslant 0$.

Now $f*g(1)=1$ by Thm. 2.12, so the case $\alpha=0$ is proved.

If $\alpha>0$, then

$$f*g(p^\alpha) = \sum_{d|p^\alpha} f(d)\,g(p^\alpha/d) = \sum_{i=1}^{\alpha} f(p^i)\,g(p^{\alpha-i})$$

$f(p^i)=0$ for $i>2$

$$= 1\cdot g(p^\alpha) + (-1-p)\,g(p^{\alpha-1}) + p\,g(p^{\alpha-2})$$

the divisors of p^k are $1, p, p^2, \dots, p^k$

$$= \sum_{i=0}^{\alpha} p^i + (-1-p)\sum_{i=0}^{\alpha-1} p^i + p\sum_{i=0}^{\alpha-2} p^i$$

$$\frac{p^{\alpha+1}-1}{p-1} + (-1-p)\cdot\frac{(p^\alpha-1)}{p-1} + \frac{p\,(p^{\alpha-1}-1)}{p-1}$$

$$= \frac{p^{\alpha+1} - p^\alpha + 1 - p^{\alpha+1} + p + p^\alpha - p}{p-1} = 0 = I(p^\alpha),$$

and we're done. (Check: this works even for $\alpha=2$.) $\square$