Homework Assignment #8: Due Monday, December 11

In this assignment, we evaluate the Riemann zeta function at several places.

1. Evaluate $\zeta(0)$. Hint: the formula for $\zeta(s)$ from the bottom of page 4 of the notes from Monday, 11/27 should help. (Recall, as noted just above that formula, that the integral that appears there is entire in s.)

2. Show that $\zeta(-2n) = 0$ for $n \in \mathbb{Z}_+$. (The negative, even integers are called the *trival zeroes* of $\zeta(s)$. There are other zeroes that are not so trivial; perhaps you've heard of them.)

3. Evaluate $\zeta(2)$. Hint: as stated in class, every sufficiently nice function of period 1 has a Fourier series expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n(f) e^{2\pi i n x} \tag{$*$}$$

   where

$$c_n(f) = \int_0^1 f(x) e^{-2\pi i n x}\, dx.$$

   FACT: if f is defined on the interval $[-1/2, 1/2]$ by $f(x) = x^2$, and is extended periodically (that is, the graph of f on $[-1/2, 1/2]$ is copied and then pasted down n units to the right and n units to the left of itself, for each positive integer n), then f is "sufficiently nice" in the above sense. (You might want to sketch the graph of f, if it helps.) Now actually *compute* the Fourier series for this particular function f. To this end, you may use the integration-by-parts formula

$$\int x^2 e^{\alpha x}\, dx = \begin{cases} \frac{x^3}{3} + C & \text{if } \alpha = 0, \\ \frac{e^{\alpha x}\left(\alpha^2 x^2 - 2\alpha x + 2\right)}{\alpha^3} + C & \text{if } \alpha \neq 0. \end{cases}$$

   (This works for complex as well as real α.) You may also want to use the fact that $e^{\pi i n} = (-1)^n$ for $n \in \mathbb{Z}$.

   Finally, once you've computed your Fourier series, apply $(*)$ with a suitable value of x.

4. Use the previous exercise, together with the functional equation (derived in class) for $\zeta(s)$, to evaluate $\zeta(-1)$.