Homework Assignment #7: SOLUTIONS

In this assignment, we derive some properties of the gamma function. (Some of these are given, without proof, in Apostol Section 12.2.)

1. Prove that $\dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = \displaystyle\int_0^1 u^{x-1}(1-u)^{y-1}\,du$, for $\mathrm{Re}(x), \mathrm{Re}(y) > 0$.

   SOLUTION: First note that, by the substitution $t = r^2$, we have

$$\Gamma(s) = \int_0^\infty e^{-t}\,t^s\,\frac{dt}{t} = \int_0^\infty e^{-r^2}\,r^{2s}\,\frac{2r\,dr}{r^2} = 2\int_0^\infty e^{-r^2}\,r^{2s-1}\,dr.$$

   So, making the suggested substitution $u = \cos^2\theta$,

$$\begin{aligned}
&\Gamma(x+y)\int_0^1 u^{x-1}(1-u)^{y-1}\,du \\
&= \left(2\int_0^\infty e^{-r^2}\,r^{2x+2y-1}\,dr\right)\int_{\pi/2}^0 (\cos\theta)^{2(x-1)}\,(1-\cos^2\theta)^{2(y-1)}\,(-2\cos\theta\sin\theta\,d\theta) \\
&= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^\infty e^{-r^2}\,r^{2x+2y-1}\cos^{2x-1}\theta\,\sin^{2y-1}\theta\,dr\,d\theta \\
&= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^\infty e^{-r^2}\,(r\cos\theta)^{2x-1}\,(r\sin\theta)^{2y-1}\,r\,dr\,d\theta.
\end{aligned}$$

   Now switch to Cartesian coordinates: put $p = r\cos\theta$ and $q = r\sin\theta$. (We can't use the usual x and y because they are already in use.) Then $r\,dr\,d\theta = dp\,dq$, so we get

$$\begin{aligned}
\Gamma(x+y)\int_0^1 u^{x-1}(1-u)^{y-1}\,du &= 4\int_{q=0}^\infty\int_{p=0}^\infty e^{-(p^2+q^2)}\,p^{2x-1}\,q^{2y-1}\,dp\,dq \\
&= \left(2\int_{p=0}^\infty e^{-p^2}\,p^{2x-1}\,dp\right)\left(2\int_{q=0}^\infty e^{-q^2}\,q^{2y-1}\,dq\right) = \Gamma(x)\Gamma(y).
\end{aligned}$$

2. Use the previous exercise to evaluate $\Gamma(1/2)$.

   SOLUTION: By the previous exercise, and (again) the substitution $u = \cos^2\theta$,

$$\begin{aligned}
\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} &= \int_0^1 u^{-1/2}(1-u)^{-1/2}\,du \\
&= \int_{\pi/2}^0 \cos^{-1}\theta\,\sin^{-1}\theta\,(-2\cos\theta\sin\theta\,d\theta) = 2\int_0^{\pi/2} d\theta = \pi.
\end{aligned}$$

   Since $\Gamma(1) = 1$ and since $\Gamma(1/2)$ is, by its definition, clearly positive, we conclude that

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

3. Prove that, for $\mathrm{Re}(s) > 0$, $\Gamma(s)\Gamma(s+1/2) = \sqrt{\pi}2^{1-2s}\Gamma(2s)$ (this is the so–called duplication formula for the gamma function). Hint: use part (a) above, and a change of variable, to show $\mathrm{B}(1/2, s) = \int_{-1}^{1}(1-v^2)^{s-1}\,dv$. Now put $v = 2w-1$.

SOLUTION: We have, by the substitution $u = v^2$,

$$\frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(s+\frac{1}{2})} = \int_0^1 u^{-1/2}\,(1-u)^{s-1}\,du = \int_0^1 v^{-1}\,(1-v^2)^{s-1}\,(2v\,dv) = 2\int_0^1 (1-v^2)^{s-1}\,dv$$
$$= \int_{-1}^{1}(1-v^2)^{s-1}\,dv,$$

the last step because the integrand is an even function of v. Putting $v = 2w-1$ gives

$$\frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(s+\frac{1}{2})} = 2\int_0^1 (4(w-w^2))^{s-1}\,dw = 2^{2s-1}\int_0^1 w^{s-1}(1-w)^{s-1}\,dw = 2^{2s-1}\frac{\Gamma(s)\Gamma(s)}{\Gamma(2s)}.$$

Doing some algebra yields

$$\Gamma(s)\Gamma(s+\tfrac{1}{2}) = 2^{1-2s}\Gamma(\tfrac{1}{2})\Gamma(2s),$$

since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we get the stated result.

4. For this exercise you should recall that, for functions $f, g : \mathbb{R} \to \mathbb{C}$, we define the *convolution* $f * g$ of f and g to be the function on $\mathbb{R}$ defined by

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy,$$

for all x such the integral exists. Also, we say that a set S of functions on $\mathbb{R}$ is a *convolution semigroup* if $f * g \in S$ whenever $f, g \in S$. Show that, if we define a function f_p on $\mathbb{R}$, for each complex number p with $\mathrm{Re}(p) > 0$, by

$$f_p(x) = \begin{cases} 0 & \text{if } x \le 0, \\ \dfrac{x^{p-1}e^{-x}}{\Gamma(p)} & \text{if } x > 0, \end{cases}$$

then the set

$$S = \{f_p \colon p \in \mathbb{C},\ \mathrm{Re}(p) > 0\}$$

is a convolution semigroup.

SOLUTION: We have

$$f_p * f_q(x) = \int_{-\infty}^{\infty} f_p(x-y)g_q(y)\,dy.$$

Now $f_p(x-y) = 0$ if $x - y \le 0$, and $g_q(y) = 0$ if $y \le 0$. So the integral becomes an integral over those y for which $x - y > 0$ and $y > 0$. In other words, the integral is over the domain $D_x = \{y \in \mathbb{R} \colon y < x \text{ and } y > 0\}$. Note that D_x is empty if $x \le 0$. Consequently,

$$f_p * f_q(x) = 0$$

if $x \le 0$. On the other hand, if $x > 0$, then $D_x = (0, x)$, so in this case

$$f_p * f_q(x) = \int_0^x f_p(x-y) g_q(y)\, dy = \frac{1}{\Gamma(p)\Gamma(q)} \int_0^x (x-y)^{p-1} e^{-(x-y)}\, y^{q-1} e^{-y}\, dy$$
$$= \frac{e^{-x}}{\Gamma(p)\Gamma(q)} \int_0^x (x-y)^{p-1}\, y^{q-1}\, dy.$$

Make the change of variable $y = xu$. Then $dy = x\,du$, and the domain of integration $(0, x)$ becomes $(0, 1)$. So if $x > 0$, we get

$$f_p * f_q(x) = \int_0^x f_p(x-y) g_q(y)\, dy = \frac{e^{-x}}{\Gamma(p)\Gamma(q)} \int_0^1 (x-ux)^{p-1}\, (ux)^{q-1} (x\, du)$$
$$= \frac{x^{p+q-1}\, e^{-x}}{\Gamma(p)\Gamma(q)} \int_0^1 (1-u)^{p-1}\, u^{q-1}\, du = \frac{x^{p+q-1}\, e^{-x}}{\Gamma(p)\Gamma(q)} \cdot \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{x^{p+q-1}\, e^{-x}}{\Gamma(p+q)},$$

the next-to-last step by Exercise 1 above.

To summarize, we've seen that

$$f_p * f_q(x) = \begin{cases} 0 & \text{if } x \le 0, \\ \dfrac{x^{p+q-1}\, e^{-x}}{\Gamma(p+q)} & \text{if } x > 0. \end{cases}$$

In other words, by definition of f_{p+q},

$$f_p * f_q(x) = f_{p+q}(x).$$

So, if $f_p, f_q \in S$, then $f_p * f_q \in S$ as well. So S is a convolution semigroup.