## Homework Assignment #6: Due Wednesday, November 1 (SOLUTIONS)

Please do the following exercises:

**Part A. Apostol Chapter 7, p. 156:** **Exercise 6.** If $(h,k)=1$, $k>0$, prove that there is a constant $A$ (depending on $h$ and $k$) such that, if $x \geq 2$,

$$\sum_{\substack{p\leq x \\ p\equiv h \pmod{k}}} \frac{1}{p} = \frac{1}{\varphi(k)} \log\log x + A + O\left(\frac{1}{\log x}\right).$$

**SOLUTION:** Let

$$a(n) = \begin{cases} \frac{\log p}{p} & \text{if } n \text{ is a prime } p \text{ that is } \equiv h \pmod{k}, \\ 0 & \text{if not.} \end{cases}$$

By Theorem 7.3, we have

$$A(x) = \sum_{n\leq x} a(n) = \sum_{\substack{p\leq x \\ p\equiv h \pmod{k}}} \frac{\log p}{p} = \frac{1}{\varphi(k)} \log x + R(x),$$

where $R(x) \leq K$ for some constant $K > 0$. So, by Abel summation with $f(x) = 1/\log x$ and $y = 3/2$, we have

$$\begin{aligned} \sum_{p\leq x} \frac{1}{p} &= \sum_{3/2<n\leq x} a(n)f(n) \\ &= A(x)f(x) - A(3/2)f(3/2) - \int_{3/2}^{x} A(t)f'(t)\,dt \\ &= \left(\frac{1}{\varphi(k)}\log x + R(x)\right)\cdot\frac{1}{\log(x)} - 0 + \int_2^x \left(\frac{1}{\varphi(k)}\log t + R(t)\right)\cdot\frac{1}{t\log^2 t}\,dt \\ &= \frac{1}{\varphi(k)} + O\left(\frac{1}{\log x}\right) + \frac{1}{\varphi(k)}\int_2^x \frac{dt}{t\log t} + \int_2^x \frac{R(t)\,dt}{t\log^2 t} \\ &= \frac{1}{\varphi(k)} + O\left(\frac{1}{\log x}\right) + \frac{1}{\varphi(k)}(\log\log x - \log\log 2) + \int_2^\infty \frac{R(t)\,dt}{t\log^2 t} - \int_x^\infty \frac{R(t)\,dt}{t\log^2 t}. \end{aligned}$$

The integral from 2 to $\infty$ is convergent, since

$$\int_2^\infty \left|\frac{R(t)}{t\log^2 t}\right| dt \leq K\int_2^\infty \frac{dt}{t\log^2 t} = -\frac{K}{\log t}\bigg|_2^\infty = \frac{K}{\log 2} < \infty.$$

So denote this integral by $B$. Regarding the integral from $x$ to $\infty$, we have

$$\left|\int_x^\infty \frac{dt}{t\log^2 t}\right| = -\frac{1}{\log t}\bigg|_x^\infty = \frac{1}{\log x}.$$

So we get

$$\begin{aligned}\sum_{p\le x}\frac{1}{p} &= \frac{1}{\varphi(k)}+O\left(\frac{1}{\log x}\right)+\frac{1}{\varphi(k)}(\log\log x-\log\log 2)+B+O\left(\frac{1}{\log x}\right)\\ &= \frac{1}{\varphi(k)}\log\log x+A+O\left(\frac{1}{\log x}\right),\end{aligned}$$

where

$$A=\frac{1-\log\log 2}{\varphi(k)}+B.$$

**Part B. Karamata's Tauberian Theorem.**

We're going to discuss a Tauberian Theorem that has many applications, one of which is to relate two different proofs of Dirichlet's Theorem on Primes in Progression.

A Tauberian Theorem is, generally, one that relates different "weighted sums" of a given nonnegative sequence $(b_n)$ (throughout, we assume that $n$ runs from 1 to $\infty$) to each other. In today's episode we study Karamata's Tauberian Theorem, which relates the behavior of

$$\sum_{n=1}^{\infty} b_n e^{-a_n T} \qquad \textbf{(1)}$$

as $T\to 0^+$ (where $(a_n)$ is another nonnegative sequence), to the behavior of

$$\sum_{a_n\le x} b_n \qquad \textbf{(2)}$$

as $x\to\infty$. (Note that, in **(1)**, each $b_n$ is weighted by $\exp(-a_nT)$, while in **(2)**, each $b_n$ is weighted by a 1 or a 0, depending on the relative size of $a_n$ to $x$.)

We'll need to recall the definition of the gamma function $\Gamma(s)$:

$$\Gamma(s)=\int_0^{\infty} e^{-t}\,t^s\,\frac{dt}{t}$$

for $\mathrm{Re}(s)>0$.

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**Exercise 1.**

(a) Show that the integral defining $\Gamma(s)$ converges absolutely (as a Lebesgue integral or an improper Riemann integral) for $\mathrm{Re}(s)>0$.

**SOLUTION:** Write $\sigma=\mathrm{Re}(s)$. We break up the integral into an integral from 0 to 1 plus an integral from 1 to $\infty$. Note that $e^{-t}\le 1$ on $[0,1]$. Also, since $\lim_{t\to\infty} e^{-t/2}t^{\sigma-1}=0$ for any real $\sigma$, and since $e^{-t/2}t^{\sigma-1}$ is continuous on $[1,\infty)$, there must be a constant $c$ such that

$e^{-t/2}t^{\sigma-1} < c$ for $t \geq 1$. So

$$\begin{aligned}\int_0^\infty \left|e^{-t}\,t^s\right| \frac{dt}{t} &= \int_0^1 \left|e^{-t}\,t^s\right| \frac{dt}{t} + \int_1^\infty \left|e^{-t}\,t^s\right| \frac{dt}{t} \\ &\leq \int_0^1 t^{\sigma-1}\,dt + c\int_1^\infty e^{-t/2}\,dt = \left.\frac{t^\sigma}{\sigma}\right|_0^1 + c\cdot\left.\frac{e^{-t/2}}{-t/2}\right|_1^\infty \\ &= \frac{1}{\sigma} + 2c\,e^{-1/2} < \infty.\end{aligned}$$

(b) Show that, for $\mathrm{Re}(s) > 0$,

$$\Gamma(s+1) = s\Gamma(s).$$

Hint: integrate by parts.

**SOLUTION:** Putting $u = t^s$ and $dv = e^{-t}\,dt$ gives

$$\Gamma(s+1) = \int_0^\infty e^{-t}t^s\,dt = -e^{-t}t^s\big|_0^\infty + s\int_0^\infty e^{-t}t^{s-1}\,dt = 0 + s\Gamma(s) = s\Gamma(s).$$

(c) Evaluate $\Gamma(1)$.

**SOLUTION:**

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = -e^{-t}\big|_0^\infty = e^0 = 1.$$

(d) Use mathematical induction to show that, for $n \in \mathbb{Z}_{\geq 0}$, $\Gamma(n+1) = n!$.

**SOLUTION:** The statement is true for $n = 0$ by part (c) of this exercise. So assume it's true for the integer $n$: $\Gamma(n+1) = n!$. Then by part (b) of this exercise,

$$\Gamma(n+2) = (n+1)\Gamma(n+1) = (n+1)n! = (n+1)!.$$

This is our inductive step, and we're done.

(e) Show that, for $a > 0$ and $\mathrm{Re}(s) > 0$,

$$a^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty e^{-at}\,t^s\,\frac{dt}{t}.$$

Hint: make a substitution into the integral on the right.

**SOLUTION:** putting $u = at$ gives $dt = du/a$ and $dt/t = (du/a)/(u/a) = du/u$, so

$$\int_0^\infty e^{-at}\,t^s\,\frac{dt}{t} = \int_0^\infty e^{-u}\left(\frac{u}{a}\right)^s\frac{du}{u} = a^{-s}\int_0^\infty e^{-u}\,u^s\,\frac{du}{u} = a^{-s}\Gamma(s).$$

We now have

**Theorem 1: Karamata's Tauberian Theorem.** Let $a = (a_n)$ and $b = (b_n)$ be sequences of nonnegative real numbers. If

$$\sum_{n=1}^{\infty} b_n e^{-a_n T} \sim A T^{-c} \tag{$*$}$$

as $T \to 0^+$, for some numbers $A, c > 0$, then

$$\sum_{a_n \le x} b_n \sim \frac{Ax^c}{\Gamma(c+1)}$$

as $x \to \infty$.

(Remark: recall that we say $f(y) \sim g(y)$ as $y \to a$ if $\lim_{y \to a}(f(y)/g(y)) = 1$.)

*Proof.* First, we need the result of the following exercise:

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**Exercise 2.** Show that, for any polynomial $P(z)$ on $[0, 1]$,

$$\sum_{n=1}^{\infty} b_n e^{-a_n T} P\left(e^{-a_n T}\right) \sim \frac{A T^{-c}}{\Gamma(c)} \int_0^{\infty} e^{-t} P\left(e^{-t}\right) t^c \, \frac{dt}{t}. \tag{$**$}$$

Some hints:

(a) Rewrite the right hand side of $(*)$ using (✸).

**SOLUTION:**

$$AT^{-c} = \frac{A}{\Gamma(c)} \int_0^{\infty} e^{-tT} t^c \, \frac{dt}{t}.$$

(b) Let $M$ be a nonnegative integer. Replace $T$ by $T(1+M)$ in your result from part (a) of this exercise, to show that $(**)$ holds for the polynomial $P(z) = z^M$.

**SOLUTION:** we have, by assumption,

$$\sum_{n=1}^{\infty} b_n e^{-a_n T(1+M)} \sim A\big[T(1+M)\big]^{-c} = \big(AT^{-c}\big)(1+M)^{-c} = \frac{AT^{-c}}{\Gamma(c)} \int_0^{\infty} e^{-t(1+M)} t^c \, \frac{dt}{t}.$$

Rewriting gives

$$\sum_{n=1}^{\infty} b_n e^{-a_n T} \big(e^{-a_n T}\big)^M \sim \frac{AT^{-c}}{\Gamma(c)} \int_0^{\infty} e^{-t(1+M)} t^c \, \frac{dt}{t} = \frac{AT^{-c}}{\Gamma(c)} \int_0^{\infty} e^{-t} \big(e^{-t}\big)^M \, \frac{dt}{t}.$$

In other words, $(**)$ is true for $P(z) = z^M$.

(c) Deduce the result $(**)$ for *any* polynomial $P$ on $[0, 1]$ from what you showed in part (b) of this exercise.

**SOLUTION:** Multiply both sides of the result from part (b) by a constant $c_M$, say, and sum from $M = 0$ to $M = D$, to get the desired result for any polynomial of degree $D$. (It's easy to show that, if $f(T) \sim pT^{-c}$ and $g(T) \sim qT^{-c}$ for constants $p$ and $q$, then $f(T) + g(T) \sim (p+q)T^{-c}$.)

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Now by **limiting arguments** (see Exercise 3 below), we can replace the polynomial $P$ in **(∗∗)** by any piecewise continuous function on $[0,1]$, and in particular by

$$P(z) = \begin{cases} 0 & \text{if } 0 \le z < e^{-1}; \\ z^{-1} & \text{if } e^{-1} \le z \le 1. \end{cases}$$

Putting this $P$ into **(∗∗)** gives

$$\sum_{e^{-1} \le e^{-a_n T} \le 1} b_n e^{-a_n T} \left(e^{a_n T}\right) \sim \frac{A\,T^{-c}}{\Gamma(c)} \int_{e^{-1} \le e^{-t} \le 1} e^{-t} \left(e^{t}\right) t^c \, \frac{dt}{t},$$

which may be rewritten

$$\sum_{0 \le a_n \le 1/T} b_n \sim \frac{A\,T^{-c}}{\Gamma(c)} \int_0^1 t^c \, \frac{dt}{t} = \frac{A\,T^{-c}}{c\Gamma(c)} = \frac{A\,T^{-c}}{\Gamma(c+1)}.$$

(The last step follows from **(🍂)**.) Putting $x = 1/T$ gives the desired result.

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**Exercise 3.** Say something about the **limiting arguments** alluded to in the above proof. You don't need to fill in all the details, but at least carefully state some theorem or other result, and explain why that result applies here.

**SOLUTION:** By Stone-Weierstrass, any piecewise continuous function on $[0,1]$ can be uniformly approximated arbitrarily well by a polynomial there. This is enough here, although it takes a bit of work to fill in all the details.

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Back to Dirichlet's theorem. Recall that our proof of this theorem amounted to showing that

$$(➡) \qquad \sum_{\substack{p \le x \\ p \equiv h \ (\mathrm{mod}\ k)}} \frac{\log p}{p} \sim \frac{1}{\phi(k)} \log x$$

as $x \to \infty$. (Certainly Dirichlet's Theorem follows from this.) See Theorem 7.3 in Apostol, where a somewhat stronger version of **(➡)**, involving a big oh term, is given. However the version given here suffices, and is more appropriate for the following discussion.

Some other proofs of Dirichlet's Theorem entail showing that

$$\sum_{\substack{p=1 \\ p \equiv h \ (\mathrm{mod}\ k)}}^{\infty} \frac{\log p}{p^s} \sim \frac{1}{\varphi(k)(s-1)} \qquad (⬆)$$

as $s \to 1^+$. The idea of the following exercise is to show that the two approaches to Dirichlet's Theorem are closely related.

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**Exercise 4.**

(a) Explain why (⬆) directly implies Dirichlet's Theorem.

**SOLUTION:** The right side of (⬆) approaches $+\infty$ as $s \to 1^+$, so the left side must too, meaning there are infinitely many terms in the sum. Hence Dirichlet's Theorem.

(b) Use Karamata's Tauberian Theorem to show that (⬆) implies (➡). Hint: in (⬆), put $s = 1 + T$.

**SOLUTION:** Putting $s = 1 + T$ into (⬆) gives

$$\sum_{\substack{p=1 \\ p \equiv h \pmod{k}}}^{\infty} \frac{\log p}{p} p^{-T} \sim \frac{T^{-1}}{\varphi(k)}$$

as $T \to 0^+$. Rewriting gives

$$\sum_{n=1}^{\infty} b_n \, e^{-a_n T} \sim \frac{T^{-1}}{\varphi(k)},$$

where

$$b(n) = \begin{cases} \frac{\log p}{p} & \text{if } n \text{ is a prime } p \text{ that is } \equiv h \pmod{k}, \\ 0 & \text{if not,} \end{cases}$$

and $a(n) = \log n$. Then by Karamata's Theorem,

$$\sum_{a_n \leq x} b_n \sim \frac{x}{\varphi(k)\Gamma(1+1)} = \frac{x}{\varphi(k)}$$

as $x \to \infty$; that is,

$$\sum_{\substack{\log n \leq x \\ n = p \equiv h \pmod{k}}} \frac{\log p}{p} \sim \frac{x}{\varphi(k)}$$

as $x \to \infty$. Replacing $x$ by $\log x$ gives

$$\sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} \frac{\log p}{p} \sim \frac{\log x}{\varphi(k)}$$

as $x \to \infty$.

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Remark: a converse to Karamata's Theorem exists, and may be used to show that, conversely, (➡) implies (⬆).