Homework Assignment #6: Due Wednesday, November 1

Please do the following exercises:

Part A. Apostol Chapter 7, p. 156: Exercise 6.

Part B. Karamata's Tauberian Theorem.

We're going to discuss a Tauberian Theorem that has many applications, one of which is to relate two different proofs of Dirichlet's Theorem on Primes in Progression.

A Tauberian Theorem is, generally, one that relates different "weighted sums" of a given nonnegative sequence (b_n) (throughout, we assume that n runs from 1 to ∞) to each other. In today's episode we study Karamata's Tauberian Theorem, which relates the behavior of

$$\sum_{n=1}^{\infty} b_n \mathrm{e}^{-a_n T} \qquad \textbf{(1)}$$

as $T \to 0^+$ (where (a_n) is another nonnegative sequence), to the behavior of

$$\sum_{a_n \le x} b_n \qquad \textbf{(2)}$$

as $x \to \infty$. (Note that, in **(1)**, each b_n is weighted by $\exp(-a_n T)$, while in **(2)**, each b_n is weighted by a 1 or a 0, depending on the relative size of a_n to x.)

We'll need to recall the definition of the gamma function $\Gamma(s)$:

$$\Gamma(s) = \int_0^\infty \mathrm{e}^{-t}\, t^s \, \frac{dt}{t}$$

for $\mathrm{Re}(s) > 0$.

Exercise 1.

(a) Show that the integral defining $\Gamma(s)$ converges absolutely (as a Lebesgue integral or an improper Riemann integral) for $\mathrm{Re}(s) > 0$.

(b) Show that, for $\mathrm{Re}(s) > 0$,

$$\Gamma(s+1) = s\Gamma(s).$$

Hint: integrate by parts.

(c) Evaluate $\Gamma(1)$.

(d) Use mathematical induction to show that, for $n \in \mathbb{Z}_{\ge 0}$, $\Gamma(n+1) = n!$.

(e) Show that, for $a > 0$ and $\mathrm{Re}(s) > 0$,

$$a^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{e}^{-at}\, t^s \, \frac{dt}{t}.$$

Hint: make a substitution into the integral on the right.

We now have

Theorem 1: Karamata's Tauberian Theorem. Let $a = (a_n)$ and $b = (b_n)$ be sequences of nonnegative real numbers. If

$$\sum_{n=1}^{\infty} b_n \mathrm{e}^{-a_n T} \sim A T^{-c} \qquad (*)$$

as $T \to 0^+$, for some numbers $A, c > 0$, then

$$\sum_{a_n \le x} b_n \sim \frac{A x^c}{\Gamma(c+1)}$$

as $x \to \infty$.

(Remark: recall that we say $f(y) \sim g(y)$ as $y \to a$ if $\lim_{y \to a}(f(y)/g(y)) = 1$.)

Proof. First, we need the result of the following exercise:

Exercise 2. Show that, for any polynomial $P(z)$ on $[0,1]$,

$$\sum_{n=1}^{\infty} b_n \mathrm{e}^{-a_n T} P\left(\mathrm{e}^{-a_n T}\right) \sim \frac{A T^{-c}}{\Gamma(c)} \int_0^{\infty} \mathrm{e}^{-t} P\left(\mathrm{e}^{-t}\right) t^c \, \frac{dt}{t}. \qquad (**)$$

Some hints:

(a) Rewrite the right hand side of $(*)$ using ().

(b) Let M be a nonnegative integer. Replace T by $T(1+M)$ in your result from part (a) of this exercise, to show that $(**)$ holds for the polynomial $P(z) = z^M$.

(c) Deduce the result $(**)$ for *any* polynomial P on $[0,1]$ from what you showed in part (b) of this exercise.

Now by **limiting arguments** (see Exercise 3 below), we can replace the polynomial P in $(**)$ by any piecewise continuous function on $[0,1]$, and in particular by

$$P(z) = \begin{cases} 0 & \text{if } 0 \le z < \mathrm{e}^{-1}; \\ z^{-1} & \text{if } \mathrm{e}^{-1} \le z \le 1. \end{cases}$$

Putting this P into $(**)$ gives

$$\sum_{e^{-1} \le e^{-a_n T} \le 1} b_n \mathrm{e}^{-a_n T} \left(\mathrm{e}^{a_n T}\right) \sim \frac{A T^{-c}}{\Gamma(c)} \int_{e^{-1} \le e^{-t} \le 1} \mathrm{e}^{-t} \left(\mathrm{e}^{t}\right) t^c \, \frac{dt}{t},$$

which may be rewritten

$$\sum_{0 \le a_n \le 1/T} b_n \sim \frac{A T^{-c}}{\Gamma(c)} \int_0^1 t^c \, \frac{dt}{t} = \frac{A T^{-c}}{c\Gamma(c)} = \frac{A T^{-c}}{\Gamma(c+1)}.$$

(The last step follows from ( ).) Putting $x = 1/T$ gives the desired result.

Exercise 3. Say something about the **limiting arguments** alluded to in the above proof. You don't need to fill in all the details, but at least carefully state some theorem or other result, and explain why that result applies here.

Back to Dirichlet's theorem. Recall that our proof of this theorem amounted to showing that

$$\sum_{\substack{p \le x \\ p \equiv h (\mathrm{mod}\ k)}} \frac{\log p}{p} \sim \frac{1}{\phi(k)} \log x \qquad (\ )$$

as $x \to \infty$. (Certainly Dirichlet's Theorem follows from this.) See Theorem 7.3 in Apostol, where a somewhat stronger version of ( ), involving a big oh term, is given. However the version given here suffices, and is more appropriate for the following discussion.

Some other proofs of Dirichlet's Theorem entail showing that

$$\sum_{\substack{p=1 \\ p \equiv h (\mathrm{mod}\ k)}}^{\infty} \frac{\log p}{p^s} \sim \frac{1}{\varphi(k)(s-1)} \qquad (\ )$$

as $s \to 1^+$. The idea of the following exercise is to show that the two approaches to Dirichlet's Theorem are closely related.

Exercise 4.

(a) Explain why ( ) directly implies Dirichlet's Theorem.

(b) Use Karamata's Tauberian Theorem to show that ( ) implies ( ). Hint: in ( ), put $s = 1 + T$.

Remark: a converse to Karamata's Theorem exists, and may be used to show that, conversely, ( ) implies ( ).