Homework Assignment #2: Due Wednesday, September 20

Throughout, n and d are positive integers.

Please read Apostol, sections 2.1–2.14. Then please do the following exercises:

Part I. Apostol Chapter 2 (pp. 46–51): Exercises 3, 4, 5, 14, 29.

Part II. Let $\delta(n)$ denote the number of positive divisors of n.

1. (a) Express $\delta(n)$ in the form

$$\sum_{d|n} f(n)$$

for an appropriate function f.

(b) Prove that

$$\sum_{d|n} \mu(d)\delta(n/d) = 1.$$

(c) Prove that

$$\sum_{d|n} \log d = \frac{\delta(n)}{2} \log n.$$

(Here and throughout, log denotes the natural logarithm.) Note: this does not depend on part (a) or (b) of this problem.

(d) Using parts (a,b,c) above, prove that

$$\log n = -\sum_{d|n} \mu(d)\delta(n/d) \log d.$$

2. (a) Express δ in the form $\delta = u * u$ for an appropriate multiplicative function u.

(b) Use the previous part of this problem to conclude that δ is multiplicative.

(c) Prove that

$$\sum_{d|n} \delta(d)^3 = \left(\sum_{d|n} \delta(d)\right)^2.$$

Hint: it suffices to show that this is true for n a power of a prime. (Explain why.)

Part III.

1. Show that $\mathcal{A} = \{\text{arithmetic functions}\}$ is a commutative ring with unity, under addition and Dirichlet multiplication.

2. Show that $\mathcal{A}$ is an integral domain. Hint: given a nonzero arithmetic function f, let

$$z_f = \min\{n \in \mathbb{Z}_+ | f(n) \neq 0\}.$$

3. Food for thought (you don't need to turn this in, but it might be a term project, or part of a term project): Is $\mathcal{A}$ a unique factorization domain? A principal ideal domain? A Euclidean domain?